COUNTING RANDOM k-SAT NEAR THE SATISFIABILITY THRESHOLD

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ABSTRACT. We present efficient counting and sampling algorithms for random *k*-SAT when the clause density satisfies $\alpha \leq \frac{2^k}{\text{poly}(k)}$. In particular, the exponential term 2^k matches the satisfiability threshold $\Theta(2^k)$ for the existence of a solution and the (conjectured) algorithmic threshold $2^k (\ln k)/k$ for efficiently finding a solution. Previously, the best-known counting and sampling algorithms required far more restricted densities $\alpha \leq 2^{k/3}$ [HWY23]. Notably, our result goes beyond the lower bound $d \geq 2^{k/2}$ for worst-case *k*-SAT with bounded-degree *d* [BGG⁺19], showing that for counting and sampling, the average-case random *k*-SAT model is computationally much easier than the worst-case model.

At the heart of our approach is a new refined analysis of the recent novel coupling procedure by [WY24], utilizing the structural properties of random constraint satisfaction problems (CSPs). Crucially, our analysis avoids reliance on the 2-tree structure used in prior works, which cannot extend beyond the worst-case threshold $2^{k/2}$. Instead, we employ a witness tree similar to that used in the analysis of the Moser-Tardos algorithm [MT10] for the Lovász Local lemma, which may be of independent interest. Our new analysis provides a universal framework for efficient counting and sampling for random atomic CSPs, including, for example, random hypergraph colorings. At the same time, it immediately implies as corollaries several structural and probabilistic properties of random CSPs that have been widely studied but rarely justified, including replica symmetry and non-reconstruction.

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1. INTRODUCTION

Random constraint satisfaction problems (CSPs) have attracted lots of attention in both computer science and statistical physics in recent years. Typically, a random CSP consists of a set of $m = \alpha n$ constraints imposed on n variables with finite domains, where the constraints are randomly generated according to a specific rule and $\alpha > 0$ is a constant representing the density of the instance. The primary goal is to find a feasible solution satisfying all constraints or more generally, optimize a random objective function specified by the constraints. Usually, with sufficiently small α , the problem is easy and can be solved in polynomial time, while as α grows, the problem becomes computationally hard at a certain point. It is, thus, important to understand the computational complexity of random CSPs with respect to the density α .

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Perhaps the most notable and important example of random CSPs in computer science is the random k-SAT problem. Let v_1, \ldots, v_n be n Boolean variables taking values in {True, False}. We construct a k-SAT formula $\Phi = \Phi(k, n, m = \lfloor \alpha n \rfloor)$ by selecting $m = \lfloor \alpha n \rfloor$ clauses independently where each clause has size k and is obtained by selecting each literal independently and uniformly at random from $\{v_1, \ldots, v_n, \neg v_1, \ldots, \neg v_n\}$. The fundamental problem for random k-SAT is to understand for what value of α , a solution (i.e., satisfying assignment of variables) exists and can be found efficiently by an algorithm. In recent years, significant progress has been made toward these problems. For random k-SAT, numerical experiments and heuristic arguments [MPZ02, MMZ05] support the *Satisfiability Conjecture*, which posits the existence of a threshold $\alpha_{sat} = \alpha_{sat}(k) = 2^k \ln 2 - (1 + \ln 2)/2 + o_k(1)$ that can be described explicitly, such that for a random k-SAT instance $\Phi(k, n, m = \lfloor \alpha n \rfloor)$,

(1)
$$\lim_{n \to \infty} \Pr\left[\Phi(k, n, m) \text{ is satisfiable}\right] = \begin{cases} 1, & \alpha > \alpha_{\text{sat}}; \\ 0, & \alpha < \alpha_{\text{sat}}. \end{cases}$$

Building on a long line of works [KKKS98, FB99, AM02, AP03, Coj14, DSS22], Ding, Sly, and Sun [DSS22] finally established (1) for sufficiently large k, confirming the prediction arising from statistical physics. For other canonical random CSPs, such as random k-uniform hypergraph q-colorings, the satisfiability threshold for the existence of a coloring remains unclear, with the current best upper and lower bounds differing by an additive $\ln 2 + o_q(1)$ factor [HM08, ACG19].

The next question is to design an efficient algorithm to find a solution at densities below the satisfiability threshold where a solution exists with high probability. It is tempting to hope that whenever a solution exists (with high probability) one can also find it in polynomial time. However, the bestknown algorithm for random *k*-SAT, developed by Coja-Oghlan [Coj10], works for densities up to $(1 - o_k(1))2^k(\ln k)/k$, leaving a gap of $\Theta(\log k/k)$ to the satisfiability threshold. Meanwhile, with the hope of approaching the satisfiability threshold α_{sat} , statistical physicists have developed sophisticated and novel message-passing algorithms (the *cavity method*) such as belief propagation, survey propagation, and their advanced versions. While numerical experiments suggest that these algorithms perform pretty well for small values of *k*, it has been rigorously proved that in general they fail beyond the density $2^k(\ln k)/k$ [Het16, CO17].

It has been noted that the density $2^k (\ln k)/k$ marks a phase transition in the geometry of the solution space; see Figure 1 for an illustration. When the density α is below $2^k (\ln k)/k$, the solution space consists of a single giant component that contains almost all solutions and any two solutions from the component can be connected by a path of solutions such that adjacent pairs have Hamming distance o(n). Meanwhile, once the density α goes slightly beyond $2^k (\ln k)/k$, the set of solutions is divided into exponentially many small clusters such that each cluster is well-connected, but any two distinct clusters are $\Omega(n)$ away in Hamming distance. In this regime, a random solution contains many frozen variables with high probability and there exist long-range correlations between variables [AC08]. For this reason, the threshold $\alpha_{clust} \approx 2^k (\ln k)/k$ is called the *clustering threshold*.

Such long-range correlations and the emergence of frozen variables in the clustering phase can be characterized by the *overlap gap property*, which is the barrier for a large family of popular algorithms, including local search algorithms and message-passing algorithms like belief and survey propagation. Thus, the algorithmic threshold for the searching problem is conjectured to coincide with the clustering threshold. This has been partially established by the recent work [BH22] of Bresler and Huang who proved that the class of *low degree polynomial algorithms* fail at density $4.91(2^k (\ln k)/k)$, by establishing a stronger version of the overlap gap property. We refer to the survey of Gamarnik [Gam21] for more discussions on the overlap gap property and its implication on the intractability of random CSPs.

In this paper, we go beyond searching a solution for random *k*-SAT and consider the even harder problems of *sampling* a uniformly random solution and *counting* the number of solutions. Sampling and counting are crucial subroutines for many other statistical and computational tasks such as inference, testing, or prediction. There have been several recent works on this topic [MS07, GGGY21, HWY23, CGG⁺24]. For random *k*-SAT, the current best algorithm is given by He, Wu, and Yang [HWY23], who presented an efficient algorithm for almost uniform sampling random *k*-SAT solutions up to densities

 $\alpha \leq 2^{k/3}$. Yet, there remains a huge exponential gap between this bound and the clustering threshold $\alpha_{\text{clust}} \approx 2^k (\ln k)/k$ for searching algorithms.

Meanwhile, message-passing algorithms such as belief and survey propagation from the cavity method are naturally inference algorithms for estimating marginal probabilities and are believed to work all the way up to the clustering threshold. Given the close relationship between inference and counting/sampling, this seems to suggest that counting and sampling may also be tractable up to the clustering threshold, which is the conjectured threshold for searching algorithms. Therefore, it is natural to ask the following question:

For random *k*-SAT, are counting and sampling tractable up to the algorithmic threshold for searching?

1.1. Counting and sampling for random *k*-SAT. We establish the tractability of approximate counting and almost uniform sampling for random *k*-SAT up to densities $\alpha \leq 2^k/\text{poly}(k)$. In particular, the exponential term 2^k matches the satisfiability and algorithmic threshold for random *k*-SAT and extends beyond the lower bound for worst-case *k*-SAT instances (see Remark 1.5 for more discussion).

The random k-SAT model is formally defined as follows.

Definition 1.1 (random *k*-SAT formulas). For $k \ge 3$, define $\Phi(k, n, m)$ as the law of the *k*-SAT formula chosen uniformly at random from all *k*-SAT formulas with *n* variables and *m* constraints.

Specifically, the random *k*-SAT formula $\Phi = (V, C) \sim \Phi(k, n, m)$ is generated as follows:

- The variable set is defined as $V = \{v_1, v_2, \dots, v_n\}$.
- The constraint set is defined as $C = \{c_1, c_2, \dots, c_m\}$, where each constraint c_i consists of exactly k literals $\ell_{i,1}, \ell_{i,2}, \dots, \ell_{i,k}$, with each literal $\ell_{i,j}$ chosen uniformly at random from all 2n literals $\{v_1, v_2, \dots, v_n, \neg v_1, \neg v_2, \dots, \neg v_n\}$.

We present results for efficient (approximate) counting and (almost uniform) sampling of random k-SAT up to densities $\alpha \leq 2^k/\text{poly}(k)$, improving upon the previous best regime $\alpha \leq 2^{k/3}$ [HWY23].

Theorem 1.2 (counting and sampling random *k*-SAT solutions). There exists a universal constant $c \ge 1$ such that the following holds with high probability over the choice of the random *k*-SAT formula $\Phi = (V, C) \sim \Phi(k, n, \lfloor \alpha n \rfloor)$ where $0 < \alpha \le 2^k/k^c$.

For any $\varepsilon > 0$, there exist the following algorithms, both with running time $(n/\varepsilon)^{\text{poly}(k,\alpha)}$:

- (Counting) A deterministic algorithm that outputs \hat{Z} which is an ε -approximation of the number of solutions $Z(\Phi)$ of Φ , i.e., $(1 \varepsilon)Z(\Phi) \le \hat{Z} \le (1 + \varepsilon)Z(\Phi)$;
- (Sampling) An algorithm that outputs a random assignment $X \in {\text{True, False}}^V$ that is ε -close in total variation distance to μ_{Φ} , the uniform distribution over all solutions of Φ .

One may hope to estimate the number of solutions $Z(\Phi)$ by simply computing and outputting $\mathbb{E}[Z(\Phi)]$; this however does not provide an effective approximation algorithm. The random *k*-SAT model does not enjoy the *superconcentration property* [BCE17, CCM⁺24] where $Z(\Phi)$ concentrates around $\mathbb{E}[Z(\Phi)]$ with constant or tiny $\omega(1)$ factors with high probability, and furthermore, the standard trick for boosting poly(*n*) approximation ratios to FPTAS [SJ89] does not apply on such random instances.

We remark that while we consider random *k*-SAT in Theorem 1.2, we can easily obtain similar results for random *regular k*-SAT of variable-degree $d = k\alpha$ whose analysis would be simpler due to the absence of high-degree vertices. For the random regular hypergraph independent set problem (equivalently, random regular *monotone k*-SAT), it was shown that the Glauber dynamics for sampling is rapidly mixing at density $\alpha = O(2^k/k^2)$ [HSZ19].

Our counting and sampling algorithms apply to general random atomic CSPs. Here, we present results for random hypergraph colorings. A hypergraph $H = (V, \mathcal{E})$ is *k*-uniform if |e| = k for all $e \in \mathcal{E}$. We adopt the following definition for the random generation of *k*-uniform hypergraphs.

Definition 1.3 (Erdős-Rényi hypergraph). For $k \ge 2$, define H(k, n, m) as the uniform distribution over all *k*-uniform hypergraphs with *n* vertices and *m* distinct hyperedges.

For a hypergraph $H = (V, \mathcal{E})$, a proper hypergraph q-coloring $X \in [q]^V$ assigns one of the q colors to each $v \in V$, ensuring no hyperedge is monochromatic. We present results for efficiently counting and sampling proper q-colorings of random k-uniform hypergraphs up to densities $\alpha \leq q^k/\text{poly}(k, q)$.

Theorem 1.4 (counting and sampling random *k*-uniform hypergraph *q*-colorings). There exists a universal constant $c \ge 1$ such that the following holds with high probability over the choice of the random hypergraph $H = (V, \mathcal{E}) \sim H(k, n, \lfloor \alpha n \rfloor)$ where $\alpha \le q^k/(qk)^c$.

For any $\varepsilon > 0$, there exist the following algorithms, both with running time $(n/\varepsilon)^{\operatorname{poly}(\log q, k, \alpha)}$:

- (Counting) A deterministic algorithm that outputs \hat{Z} which is an ε -approximation of the number of proper q-colorings Z(H,q) of H, i.e., $(1 \varepsilon)Z(H,q) \le \hat{Z} \le (1 + \varepsilon)Z(H,q)$;
- (Sampling) An algorithm that outputs a random assignment $X \in [q]^V$ that is ε -close in total variation distance to μ_H , the uniform distribution over all proper q-colorings of H.

An analog of Theorem 1.4 holds for random regular hypergraph colorings as well.

Remark 1.5 (Comparison with worst-case bounded-degree CSPs). It is helpful to review the literature on counting and sampling for worst-case bounded-degree CSPs. A random *k*-SAT formula with density α has an average degree of $k\alpha$, making it comparable to a *k*-SAT formula with maximum degree $d = k\alpha$. For bounded-degree CSPs, while a solution exists and can be efficiently found when $d \leq 2^k$ [EL75, MT10] via the celebrated (algorithmic) Lovász local lemma, the problem of sampling and counting solutions becomes intractable when $d \geq 2^{k/2}$ [BGG⁺19]. A similar separation occurs for *k*-uniform hypergraph *q*-colorings: while the existence/searching problem can be solved when $d \leq q^k$, the sampling/counting problem becomes intractable at $d \geq q^{k/2}$ [GGW22]. This demonstrates that the sampling and counting problems are computationally much harder than searching for worst-case bounded-degree CSPs. In contrast, our findings demonstrate that for random CSPs, the sampling and counting problems are not significantly computationally harder than their satisfiability and algorithmic counterparts.

Remark 1.6. As mentioned, our main results Theorems 1.2 and 1.4 in fact apply to a broad family of random CSPs (see Theorem 3.2 for a formal statement). One may wonder if the counting/sampling threshold would always be close to or even match the searching threshold for general random CSPs. The answer is probably no in general. A very recent paper [EG24] by El Alaoui and Gamarnik shows that for the symmetric binary perceptron model, another important example of random CSPs, sampling a solution is intractable at *any* constant density (for two common classes of algorithms) while there are known algorithms for finding solutions at sufficiently low density.

We note that the random CSPs considered in this paper are *sparse*, in the sense that both the size of constraints and the (average) degree of variables are constant. Meanwhile, the symmetric binary perceptron model is *dense* as the constraint size and variable degrees are both linear. This leads to a huge difference in the geometry of the solution space: For the symmetric binary perceptron model at any constant density, most solutions are isolated with linear distance from each other. Therefore, it is still possible that the counting/sampling threshold is equivalent to the searching threshold, or at least somewhere near it, for general sparse random CSPs as considered in this paper.

1.2. **Constraint-wise coupling and replica symmetry.** Given the extensive focus in the literature, we primarily present our results for random *k*-SAT in this subsection; however, the results and proofs are equally applicable to random hypergraph colorings or other random atomic CSPs.

The central tool for our algorithmic results is a constraint-wise coupling developed in [WY24]. Let $\Phi = (V, C) \sim \Phi(k, n, \lfloor \alpha n \rfloor)$ be a *k*-SAT instance at density $\alpha > 0$, and let $c_0 \in C$ be an arbitrary clause. We define μ_{Φ} as the uniform distribution over all solutions to Φ , and $\mu_{\Phi \setminus c_0}$ as the uniform distribution over solutions to $\Phi \setminus c_0 := (V, C \setminus \{c_0\})$ with the clause c_0 removed from Φ . We establish the following result concerning the 1-Wasserstein distance between the two distributions μ_{Φ} and $\mu_{\Phi \setminus c_0}$.

Theorem 1.7. There exists a universal constant $c \ge 1$ such that the following holds with high probability over the choice of the random k-SAT formula $\Phi = (V, C) \sim \Phi(k, n, \lfloor \alpha n \rfloor)$ where $0 < \alpha \le 2^k/k^c$.

For any $c_0 \in C$, it holds that

$$W_1\left(\mu_{\Phi},\mu_{\Phi\setminus c_0}\right)=O(\log n),$$

where $W_1(\cdot, \cdot)$ denotes the 1-Wasserstein distance with respect to the Hamming metric. That is, there exists a coupling (X, Y) of μ_{Φ} and $\mu_{\Phi\setminus c_0}$ such that $\mathbb{E}[d_{\text{Ham}}(X, Y)] = O(\log n)$.



FIGURE 1. The heuristic diagram in [DSS22] depicts phase transitions in the geometry of the solution space of a random *k*-SAT instance as the density α increases from left to right.

Theorem 1.7 is at the heart of our counting and sampling algorithms from Theorem 1.2. Intuitively, it states that given a solution *Y* sampled from $\mu_{\Phi\setminus c_0}$ (which may violate c_0), we can flip the value of at most $O(\log n)$ variables with high probability to obtain a truth assignment *X* in such a way that *X* is a solution to Φ and is distributed uniformly at random as μ_{Φ} . We establish Theorem 1.7 by constructing a recursive coupling procedure and showing it terminates with high probability within $O(\log n)$ iterations. We then apply an LP-based algorithm introduced by Moitra [Moi19] which provides fast counting and sampling algorithms; this is also the strategy in [WY24].

The coupling result in Theorem 1.7 can be understood as a way to describe the *decay of correlation* phenomenon. In particular, it is similar to various other notions of correlation decay such as disagreement percolation for showing Gibbs uniqueness [vdBS94], recursive coupling for proving strong spatial mixing [GMP05], and especially coupling independence for showing spectral independence [CZ23, CGG⁺24]. For random CSPs like *k*-SAT, the Wasserstein distance between the original formula and the formula obtained after the removal of a single clause is reminiscent of the cavity method. However, establishing Theorem 1.7 for random *k*-SAT is quite different from those previous works on distinct models. Firstly, the *k*-SAT instance is based on hypergraphs while previous rigorous approaches for correlation decay mostly consider models defined on graphs. Secondly, the hypergraph associated with a random *k*-SAT formula contains a large maximum degree, making the analysis significantly more challenging. Finally, Theorem 1.7 considers the removal of a constraint, while previous approaches often consider flipping the value of a variable.

Before discussing our proof approach for Theorem 1.7 in Section 1.3, we first mention a few direct yet important implications of Theorem 1.7 for random k-SAT at the considered densities.

Many empirical studies in statistical physics use heuristics to predict the solution space geometry of random *k*-SAT, with the (predicted) phase transitions illustrated in Figure 1. Extensive research from both computer science and statistical physics has been dedicated to understanding these phases and their corresponding thresholds [MMZ05, DMMZ05, ART06, CP12, ZK16, BS20]. Notably, the satisfiability threshold α_{sat} has been precisely determined by [DSS22] for sufficiently large *k*. Other important thresholds, such as the clustering threshold α_{clust} (also referred to as the dynamic phase transition threshold in [KMRT⁺07]), where the solution space of a *k*-SAT fragments into an exponential number of clusters, and the condensation threshold α_{cond} , beyond which the solution space is dominated by a few large clusters, have been extensively studied but are not yet fully characterized.

One of the key tools for statistical physicists in understanding and describing the solution space geometry of random *k*-SAT are the notions of *replica symmetry* and *replica symmetry breaking* [MRTS08], first introduced by Parisi [Par79]. Informally, "replica symmetry" refers to the idea that two uniformly chosen variables are nearly independent, while "replica symmetry breaking" corresponds to the existence of extensive long-range correlations. Following [KMRT⁺07], we present the formal definition below.

Definition 1.8 (replica symmetry). Given k, α , we say that a random k-SAT model with density α is *replica symmetric* if, for $\Phi = (V, C) \sim \Phi(k, n, \lfloor \alpha n \rfloor)$, a uniform satisfying assignment $\sigma \sim \mu_{\Phi}$, and two variables $v_1, v_2 \in V$ chosen uniformly at random, the following holds:

$$\lim_{n \to \infty} |\mathbf{Pr} \left[\sigma(v_1) = \sigma(v_2) = \mathsf{True} \right] - \mathbf{Pr} \left[\sigma(v_1) = \mathsf{True} \right] \mathbf{Pr} \left[\sigma(v_2) = \mathsf{True} \right] | = 0.$$

Definition 1.8 essentially states that the events $\sigma(v_1) = \text{True}$ and $\sigma(v_2) = \text{True}$ are asymptotically independent for large *n*, therefore indicating the absence of long-range correlations in a relatively weak sense, as the typical distance between v_1 and v_2 is $\Omega(\log n)$. Replica symmetry is conjectured to hold up to the condensation threshold α_{cond} , which has been verified on a few other models; see e.g. [COKPZ17, COEJ⁺18] and the references therein. For random *soft-constraint k-*SAT, it has been shown that replica symmetry is sufficient for the success of the Belief Propagation algorithm [CMR22].

While the replica symmetry condition has been extensively discussed in the literature, this property is rarely proved rigorously. In this work, we show replica symmetry holds under the considered densities, which follows immediately from Theorem 1.7.

Theorem 1.9 (replica symmetry of random *k*-SAT). Under the condition of Theorem 1.2, the random *k*-SAT model with density α is replica symmetric.

Another important property that arises from the study of the random *k*-SAT by statistical physicists is the *(non-)reconstruction* property [MM06], which informally requires one being able to estimate the value of one variable given "far away" observations [GM07, MRT11].

For an SAT instance $\Phi = (V, C)$, we use vbl(*c*) to denote the set of variables involved in *c* for each clause $c \in C$. We let $H_{\Phi} = (V, \mathcal{E})$ denote the hypergraph where *V* is the set of variables in Φ and $\mathcal{E} = \{\text{vbl}(c) \mid c \in C\}$. For a hypergraph *H*, we use $d_H(\cdot, \cdot)$ to denote the graph-theoretic distance in *H*. Finally, for a hypergraph $H = (V, \mathcal{E})$, a vertex $v \in V$, and any $r \ge 1$, we let $\overline{B}_H(v, r) \triangleq \{u \in V \mid d_H(u, v) \ge r\}$ to denote the set of vertices in *V* with distance to *v* at least *r*. We then follow [MRT11] to give the following formal definition of the non-reconstruction property.

Definition 1.10 (non-reconstruction). Given k, α , we say that a random k-SAT model with density α is *non-reconstructible* if, for $\Phi = (V, C) \sim \Phi(k, n, \lfloor \alpha n \rfloor)$ with the uniform distribution over satisfying assignments $\mu = \mu_{\Phi}$ and the induced hypergraph $H = H_{\Phi}$, the following holds for any $v \in V$:

$$\lim_{r \to \infty} \limsup_{n \to \infty} \mathbb{E} \left[d_{\mathrm{TV}} \left(\mu_{\{\nu\} \cup \bar{B}_H(\nu, r)}, \mu_{\nu} \otimes \mu_{\bar{B}_H(\nu, r)} \right) \right] = 0.$$

We say that the model is *reconstructible* otherwise. Here, μ_v , $\mu_{\bar{B}_H(v,r)}$, $\mu_{\{v\}\cup\bar{B}_H(v,r)}$ denote the marginal distributions induced from μ on the subset of variables $\{v\}$, $\bar{B}_H(v,r)$, $\{v\}\cup\bar{B}_H(v,r)$, respectively.

The non-reconstruction property, which also reflects the absence of long-range correlations, is a stronger condition than replica symmetry [CMR22]. In fact, non-reconstructibility is a necessary condition for the rapid mixing of Glauber dynamics or other local Markov chains [BKMP05]. Furthermore, non-rigorous statistical mechanics calculations [MPZ02] and approximations up to the second order [MRT11] strongly suggest that the threshold for non-reconstruction aligns with the clustering threshold α_{clust} . However, there is no rigorous proof of this as far as we know. In this work, we demonstrate that non-reconstruction holds under the considered densities, which is again a direct corollary of Theorem 1.7.

Theorem 1.11 (non-reconstruction of random *k*-SAT). Under the condition of Theorem 1.2, the random *k*-SAT model with density α is non-reconstructible.

We further establish *looseness* of variables under the considered densities, an intuitive way of characterizing the connectivity of the solution space [AC08, CGG⁺24]; see Definition 4.12 and Theorem 4.13 for details.

1.3. **Technique Overview.** We establish all of the results mentioned above in the context of atomic CSPs. A CSP is considered atomic if each constraint is violated by exactly one assignment in its domain. Any constraint with a constant number *N* of violating configurations can be decomposed into *N* atomic constraints. Both SAT and hypergraph coloring instances fall under this category of atomic CSPs.

Our algorithm for counting and sampling atomic CSP solutions is inspired by the recently developed recursive coupling procedure for bounded-degree atomic CSPs in [WY24]. Through the novel recursive coupling procedure, [WY24] successfully dispenses with the freezing paradigm equipped by the previous approaches [Bec91, Moi19, FHY21, JPV21], therefore by passing the technical barrier and achieving a $q^k \gtrsim d^{2+o_q(1)}$ bound for counting/sampling bounded-degree CSPs, where d is the maximum degree of the instance. The freezing paradigm and its variant are also applied by all recent counting/sampling algorithms for random CSPs [GGGY21, HWY23, CGG⁺24]. Hence, we naturally circumvent this barrier for random CSPs.

However, the bound in [WY24] has an exponent of $2 + o_q(1)$, which only approaches 2 for large domain sizes and rises to ≈ 4.82 for the worst-case of atomic CSPs with Boolean domains, such as for *k*-SAT. The way we improve this exponent to 1 + o(1) is by leveraging *a new analysis of the coupling procedure based on the structural properties of random CSPs*. Random CSPs enjoy good structural properties such as large constraint expansion, as already observed and utilized in [HWY23]. Our main novelty lies in designing a new analysis for the coupling in [WY24] that takes advantage of this structural property. Technically, we replace the 2-tree witness employed in the analysis of the coupling in [WY24] by a denser witness structure, constructed similarly as the witness tree structure for analyzing the Moser-Tardos algorithm [MT10], thereby achieving the improved bound.

This refined analysis of the coupling procedure immediately establishes properties such as replica symmetry and non-reconstruction at the same density. We believe this new analysis is of independent interest and may have further applications in both bounded-degree and random CSPs.

1.4. Organization. This paper is organized as follows.

In Section 2, we formally define atomic CSPs and introduce relevant preliminaries and notations.

In Section 3, we define the main structural condition (Condition 3.1), stated generally with respect to atomic CSPs. We then claim that this condition suffices for efficient sampling and counting of solutions (Theorem 3.2), and prove that both the random k-SAT instance in Theorem 1.2 and the random hypergraph coloring instance in Theorem 1.4 satisfy this condition.

Section 4 and Section 5 together proves Theorem 3.2, which concludes the proof of Theorem 1.2 and Theorem 1.4. Specifically, Section 4 introduces the coupling in [WY24] with our new analysis and establishes Theorem 1.7. Section 5 shows how to effectively convert this coupling into an efficient counting and sampling algorithm.

In Section 4.4, we prove the properties related to the correlation decay phenomenon and the geometry of the solution space, namely Theorems 1.9, 1.11 and 4.13, using the refined analysis of the coupling developed in Section 4.

2. Preliminaries

2.1. Atomic CSPs and related notations. A constraint satisfaction problem (CSP) is described by a collection of constraints defined on a set of variables. Formally, an instance of a constraint satisfaction problem, called a *CSP formula*, is denoted by $\Phi = (V, [q], C)$. Here, V is a set of n = |V| random variables, where each random variable $v \in V$ has a finite domain [q]. The collection of local constraints is given by C, where each constraint $c \in C$ is a function defined as $c : [q]^{\text{vbl}(c)} \rightarrow \{\text{True}, \text{False}\}$ over a subset of variables denoted as $\text{vbl}(c) \subseteq V$. For any subset of constraints $\mathcal{E} \subseteq C$, denote $\text{vbl}(\mathcal{E}) \triangleq \bigcup_{c \in \mathcal{E}} \text{vbl}(c)$. An assignment $\sigma \in [q]^V$ is called *satisfying* for Φ if

$$\Phi(\sigma) \triangleq \bigwedge_{c \in C} c\left(\sigma_{\operatorname{vbl}(c)}\right) = \operatorname{True}.$$

Furthermore, we say Φ is *satisfiable* if at least one satisfying assignment to Φ exists. We use Ω_{Φ} to denote the set of satisfying assignments to Φ , and use $Z(\Phi) = |\Omega_{\Phi}|$ to denote the number of satisfying assignments to Φ .

We say a constraint $c \in C$ is defined by *atomic bad events*, or simply, *atomic*, if it is violated by exactly one configuration in $[q]^{\text{vbl}(c)}$. For an atomic constraint c, we use False(c) to denote its only violating configuration in $[q]^{\text{vbl}(c)}$. Moreover, when all constraints in C are atomic, we say Φ is atomic. It is important to note that any constraint with a constant number N of violating configurations can be decomposed into N atomic constraints. As a result, both SAT and hypergraph coloring instances fall within the category of atomic CSPs.

2.1.1. Notations for (partial) assignments. For a partial assignment $\sigma \in [q]^{\Lambda}$ specified over a subset of variables $\Lambda \subseteq V$, we use $\Lambda(\sigma) = \Lambda$ to denote the set of assigned variables in σ . For any partial

assignment σ and any $S \subseteq \Lambda(\sigma)$, we use σ_S to denote $\bigotimes_{v \in S} \sigma(v)$. We further write $\sigma_v = \sigma_{\{v\}}$ for $v \in V$.

For any two assignments σ, τ such that $\Lambda(\sigma) \cap \Lambda(\tau) = \emptyset$, we define $\sigma \wedge \tau \in [q]^{\Lambda(\sigma) \cup \Lambda(\tau)}$ as the concatenation of σ and τ such that for any $v \in \Lambda(\sigma) \cup \Lambda(\tau)$,

$$(\sigma \wedge \tau)(v) = \begin{cases} \sigma(v) & v \in \Lambda(\sigma), \\ \tau(v) & v \in \Lambda(\tau). \end{cases}$$

We will use \emptyset to specifically denote an empty assignment, distinguishing from the empty set \emptyset .

2.1.2. *Notations for events and probability measures.* We begin by specifying some notations for events and probability measures related to the CSP.

Definition 2.1 (simple notations for events). For simplicity of notation, we will use:

- a constraint $c \in C$ to denote the event that this constraint is satisfied;
- a subset of constraints $\mathcal{E} \subseteq C$ to denote the event that all constraints in \mathcal{E} are satisfied;
- a partial assignment σ to denote the event that the assignment on $\Lambda(\sigma)$ is precisely σ .

Note that under this definition, the notation $\sigma \wedge \tau$ as a concatenation of assignments defined previously is consistent with its interpretation as an event, where $\sigma \wedge \tau$ is considered as the logical "and" of the two events σ and τ .

We use \mathcal{P} to denote the uniform product distribution over the space $[q]^V$. We use $\mu = \mu_{\Phi}$ to denote the distribution over all satisfying assignments of Φ induced by \mathcal{P} , i.e.

$$\mu_{\Phi} \triangleq \mathcal{P}\left(\cdot \mid C\right).$$

 μ_{Φ} is well-defined only when Φ is satisfiable.

When the variable set *V* and the domain [q] are clear, we define the following notations for (conditional) distributions for a given set of constraints \mathcal{E} defined over *V*, and some assignment σ defined over $\Lambda(\sigma)$:

$$\mu_{\mathcal{E}} \triangleq \mathcal{P}(\cdot \mid \mathcal{E}), \quad \mu_{\mathcal{E}}^{\sigma} \triangleq \mathcal{P}(\cdot \mid \mathcal{E} \land \sigma).$$

For a probability distribution μ and some subset of variables $\Lambda \subseteq V$, we use μ_{Λ} to denote the marginal distribution induced by μ on Λ . We use commas to separate multiple subscripts; for example, we use $\mu_{\mathcal{E},\Lambda}^{\sigma}$ to denote the marginal distribution induced by $\mu_{\mathcal{E}}^{\sigma}$ on Λ .

2.1.3. *Pinned formula and pinned constraints.* For a subset of variables $\Lambda \subseteq V$ and a partial assignment $\sigma \in [q]^{\Lambda}$ specified on Λ , the pinned formula $\Phi = (V, [q], C)$ under σ , denoted by $\Phi^{\sigma} = (V^{\sigma}, [q], C^{\sigma})$, is a new CSP formula such that $V^{\sigma} = V \setminus \Lambda(\sigma)$, and the C^{σ} is obtained from C by:

- (1) replacing each original constraint $c \in C$ with the corresponding pinned constraint c^{σ} , where $vbl(c^{\sigma}) = vbl(c) \setminus \Lambda(\sigma)$ and $c^{\sigma}(\tau) = c(\tau \wedge \sigma_{\Lambda(\sigma) \cap vbl(c)})$ for any $\tau \in [q]^{vbl(c^{\sigma})}$;
- (2) removing all the resulting constraints that have already been satisfied.

Whenever a pinning σ is applied to a CSP formula $\Phi = (V, [q], C)$, we always assume that σ does not violate any constraint in C. Under this assumption, Φ^{σ} is always well-defined, and we have $\mu_{\Phi^{\sigma}} = \mu_{V \setminus \Lambda(\sigma)}^{\sigma}$. We use C^* to denote the set of all possible constraints obtained from pinning some constraint in C with a non-violating σ , including the unpinned constraints in C. Finally, for each (possibly pinned) constraint $c \in C^*$, we use c^O to denote its original unpinned constraint in C.

2.1.4. *The incidence hypergraph.* We define the underlying *incidence hypergraph* of CSP formulas as follows.

Definition 2.2 (incidence (hyper-)graphs for variables and constraints). Given a CSP formula $\Phi = (V, [q], C)$, we define two incidence (hyper-)graphs for variables and constraints respectively:

- We define $H_{\Phi} = (V, \mathcal{E})$ to be the hypergraph (with multiple edges allowed), where *V* is the set of variables in Φ , and $\mathcal{E} = {vbl(c) | c \in C}$.
- Let $G_{\Phi} = \text{Lin}(H_{\Phi})$ to denote the line graph of H_{Φ} , namely, the vertices in G_{Φ} are clauses in Φ , and two clauses c_1, c_2 are adjacent in G_{Φ} if and only if $\text{vbl}(c_1) \cap \text{vbl}(c_2) \neq \emptyset$.

In particular, when the context of Φ is clear, we say a subset of variables $V' \subseteq V$ is connected if the induced sub-graph $H_{\Phi}[V']$ is connected and say a subset of constraints $C' \subseteq C$ is connected if the induced sub-graph $G_{\Phi}[C']$ is connected.

Similar to the density of random formulas (the ratio between the number of clauses and the number of variables), we define the density of hypergraphs as the ratio between the number of hyperedges and the number of vertices.

2.2. Lovász local lemma. The Lovász local lemma is a gem in the probabilistic method of combinatorics and has inseparable connections with the solution space of CSPs [EL75]. By viewing the violation of each constraint as a bad event, the celebrated (variable framework) Lovász local lemma gives a sufficient criterion for a CSP solution to exist:

Theorem 2.3 (Erdős and Lovász [EL75]). Given a CSP formula $\Phi = (V, [q], C)$, if the following holds

(2)
$$\exists x \in (0,1)^{\mathcal{C}} \quad s.t. \quad \forall c \in \mathcal{C} : \quad \mathcal{P}\left[\neg c\right] \le x(c) \prod_{\substack{c' \in \mathcal{C} \\ \operatorname{vbl}(c) \cap \operatorname{vbl}(c') \neq \emptyset}} (1-x(c')),$$

then

$$\mathcal{P}\left[\bigwedge_{c\in C} c\right] \ge \prod_{c\in C} (1-x(c)) > 0.$$

When the condition (2) is met, the probability of any event in the uniform distribution μ over all satisfying assignments can be well approximated by the probability of that event in the product distribution. This was observed in [HSS11]:

Theorem 2.4 (Haeupler, Saha, and Srinivasan [HSS11]). *Given a CSP formula* $\Phi = (V, [q], C)$, *if* (2) holds, then for any event A that is determined by the assignment on a subset of variables $vbl(A) \subseteq V$,

$$\mathcal{P}\left[A \mid \bigwedge_{c \in C} c\right] \le \mathcal{P}(A) \prod_{\substack{c \in C \\ \operatorname{vbl}(c) \cap \operatorname{vbl}(A) \neq \emptyset}} (1 - x(c))^{-1}$$

3. Structural properties of random hypergraphs

Our strategy to prove main results in Theorem 1.2 and Theorem 1.4 is to establish that both instances possess certain structural properties that enable efficient counting and sampling algorithms. In this section, we will develop a "nice property" (formally defined in Definition 3.12) for the underlying incidence hypergraph. We claim that once this nice property holds for the hypergraph with specific parameters, we can efficiently count and sample solutions. Finally, we demonstrate that the conditions in Theorem 1.2 and Theorem 1.4 fulfill the desired property.

Recall the definition of incidence hypergraphs in Definition 2.2. We then present our claim as Theorem 3.2, with the "nice property" formally defined later in this section in Definition 3.12. We note that Condition 3.1 only makes assumptions to the hypergraph H_{Φ} , but not the atomic bad events for each constraint.

Condition 3.1 (structural condition for atomic CSPs). For the atomic CSP instance $\Phi = (V, [q], C)$, its incidence hypergraph H_{Φ} is $(k, \alpha, \varepsilon_1, \varepsilon_2, \eta, \rho, p_1, p_2)$ -nice with parameters satisfying:

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• $k \ge 30, \alpha \le q^k, \eta k \ge 4, and e(\rho k \alpha)^{\eta} = 1;$ • $\varepsilon_1 = 2\eta, p_1 = 6k^7, \varepsilon_2 = \frac{12k^5}{(1-\eta)\eta p_1} = \frac{2}{k^2(1-\eta)\eta} \le \frac{1}{k}, and p_2 = ek^2.$

Theorem 3.2 (counting/sampling for atomic CSPs). Assume $\Phi = (V, [q], C)$ is a satisfiable atomic CSP instance satisfying Condition 3.1. Then for any $\varepsilon > 0$, there exist the following algorithms, both with running time $(n/\varepsilon)^{\operatorname{poly}(\log q, k, \alpha)}$:

- (counting) A deterministic algorithm that outputs \hat{Z} which is an ε -approximation of the number of solutions $Z(\Phi)$ of Φ , i.e., $(1 - \varepsilon)Z(\Phi) \le \hat{Z} \le (1 + \varepsilon)Z(\Phi)$;
- (sampling) An algorithm that outputs a random assignment $X \in [q]^V$ that is ε -close in total variation distance to μ_{Φ} , the uniform distribution over all solutions of Φ .

Also, we claim that both conditions in Theorems 1.2 and 1.4 satisfy the desired structural properties.

Theorem 3.3. Both the random *k*-SAT instance under the condition in Theorem 1.2 and the random hypergraph coloring instance under the condition in Theorem 1.4 satisfy Condition 3.1 with high probability.

Note that the condition in Theorem 1.2 is below the explicit threshold $\alpha < 1.3836 \cdot 2^k/k$, as established in [FS96, Theorem 1.3]. Similarly, the condition in Theorem 1.4 falls below the explicit threshold $\alpha < q^{k-1} \ln k$, as given in [DFG15, Theorem 1.1]. As a result, the instances described in Theorems 1.2 and 1.4 are satisfiable with high probability under respective conditions. Consequently, Theorems 3.2 and 3.3 together imply Theorems 1.2 and 1.4.

The plan goes as follows: For the rest of this section, we formally define the structural properties and verify that Theorem 3.3 holds. Theorem 3.2 will be proved later in Sections 4 and 5.

We now start to introduce the desired structural properties of the CSP formula. Most of the properties and proofs of random CSP formulas were presented in [GGGY21, HWY23] using different parameters. Here, we will adapt the proofs to our parameters and show that all properties hold in the uniform hypergraph model as well.

Let \mathcal{H}_k be the set of all *k*-uniform hypergraphs, and $\mathcal{H}_{\leq k}$ be the set of hypergraphs where each hyperedge contains at most *k* vertices. Now we can describe the following properties for hypergraphs.

Property 3.4 (bounded maximum degree). Given a hypergraph $H = (V, \mathcal{E})$. The maximum degree $\Delta = \Delta(H)$ is at most $4k\alpha + 6 \log n$.

Property 3.5 (edge expansion). For $\eta, \rho \in (0, 1)$, we say a hypergraph $H = (V, \mathcal{E}) \in \mathcal{H}_{\leq k}$ satisfies (η, ρ) -edge expansion if for any $\ell \leq \rho |\mathcal{E}|$ and any ℓ hyperedges $e_1, \ldots, e_\ell \in \mathcal{E}$, it holds

$$\left|\bigcup_{i=1}^{\ell} e_i\right| \ge (1-\eta)k\ell.$$

Property 3.6 (bounded neighbourhood growth). For any $e \in \mathcal{E}$ and any $\ell \ge 1$, the number of connected subsets of hyperedges containing e of size ℓ is at most $n^3(p_2\alpha)^{\ell}$.

Similar to the approaches in [GGGY21, HWY23], a crucial component of our algorithm for random instances involves identifying high-degree vertices, as well as those whose marginal distributions may be significantly affected by high-degree ones. We begin with the identifying subroutine.

Definition 3.7 (high-degree vertices). Given a hypergraph $H = (V, \mathcal{E})$ with average degree d, and a subset of vertices $V' \subseteq V$, let $HD(V') \triangleq \{v \in V' \mid \deg(v) \ge p_1 \alpha\}$ denote the set of *high-degree* vertices in V'.

ŀ	Algorithm 1: IdentifyBad(V ₀) [HWY23]	
	Instance : a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with average degree <i>d</i> ;	
	Input : a set of vertices $V_0 \subseteq V$;	
	Output : a set of bad vertices $V_{\text{bad}}(V_0)$ starting from V_0 and a set of bad hyperedges $\mathcal{E}_{\text{bad}}(V_0)$;	
1	Initialize $V_{\text{bad}}(V_0) \leftarrow \text{HD}(V_0)$ and $\mathcal{E}_{\text{bad}}(V_0) = \emptyset$;	
2	while $\exists e \in \mathcal{E} \setminus \mathcal{E}_{bad}(V_0)$ such that $ e \cap V_{bad}(V_0) > \varepsilon_1 k$ do	
3	Update $V_{\text{bad}}(V_0) \leftarrow V_{\text{bad}}(V_0) \cup e$ and $\mathcal{E}_{\text{bad}}(V_0) \leftarrow \mathcal{E}_{\text{bad}}(V_0) \cup \{e\}$	
4	return $V_{\text{bad}}(V_0)$ and $\mathcal{E}_{\text{bad}}(V_0)$	

Throughout, we will use the notation $V_{\text{bad}} = V_{\text{bad}}(V)$ and $\mathcal{E}_{\text{bad}} = \mathcal{E}_{\text{bad}}(V)$ if the hypergraph $H = (V, \mathcal{E})$ is clear from the context. The set of *good vertices* $V_{\text{good}} \triangleq V \setminus V_{\text{bad}}$ and the set of *good constraints* $\mathcal{E}_{\text{good}} \triangleq \mathcal{E} \setminus \mathcal{E}_{\text{bad}}$ is defined to be the set of remaining variables/hyperedges.

Fact 3.8 (bounded-degree for good vertices). *For every good vertex* $v \in V_{\text{good}}$ *, it holds* deg(v) $\leq p_1 \alpha$.

Fact 3.9 (bounded fraction of good vertices). For every good hyperedge $e \in \mathcal{E}_{\text{good}}$, it holds $(1 - \varepsilon_1)k \le |e \cap V_{\text{good}}| \le k$.

The following structural properties are useful in our counting and sampling algorithms.

Property 3.10 (bounded number of bad vertices). Given a hypergraph $H = (V, \mathcal{E})$, for any $V_0 \subseteq V$, $|V_{\text{bad}}(V_0)| \leq 4\varepsilon_1^{-1} |\text{HD}(V_0)|$, where $V_{\text{bad}}(V_0)$ is obtained from Algorithm 1.

Property 3.11 (bounded fraction of bad hyperedges). Given a hypergraph $H = (V, \mathcal{E})$, let $\mathcal{E}_{bad} = \mathcal{E}_{bad}(V)$ be the set of bad hyperedges obtained from Algorithm 1. For any $\ell \ge \log n$ and any connected subset of hyperedges in Lin(H) of size ℓ , the number of bad hyperedges among them is at most $\varepsilon_2 \ell$.

Definition 3.12 (nice hypergraph). We say a hypergraph $H = (V, \mathcal{E})$ is $(k, \alpha, \varepsilon_1, \varepsilon_2, \eta, \rho, p_1, p_2)$ -nice if with the choice of p_1 at Definition 3.7 and ε_1 at Algorithm 1, the hypergraph H:

- is in $\mathcal{H}_{\leq k}$ and has density α where $\alpha \leq q^k$;
- satisfies bounded maximum degree defined at Property 3.4;
- satisfies (η, ρ) -constraint expansion defined at Property 3.5;
- satisfies bounded neighbourhood growth with parameter p_2 at Property 3.6;
- satisfies bounded number of bad vertices with parameter ε_1 at Property 3.10;
- satisfies bounded fraction of bad hyperedges with parameter ε_2 at Property 3.11;
- has no connected components of size $\ell \ge \log n$ in the line graph induced by bad hyperedges.

The following key lemma shows that if all parameters $(k, \alpha, \varepsilon_1, \varepsilon_2, \eta, \rho, p_1, p_2)$ satisfy Condition 3.1, then the underlying incidence hypergraphs of random *k*-SAT formulas is nice with probability 1-o(1/n).

Lemma 3.13. For any fixed parameters $k, \alpha, \eta, \rho, \varepsilon_1, p_1, \varepsilon_2, p_2$ where $\eta k \ge 4$, $e(\rho k \alpha)^{\eta} \le 1$, $\varepsilon_1 = 2\eta$, $6k^5 \le p_1 \le e^{k-2}\alpha$, $\varepsilon_2 = 12k^5/((1-\eta)\eta p_1)$ and $p_2 = ek^2$, with probability 1 - o(1/n) over the choice of random k-SAT formulas $\Phi = \Phi(k, n, m)$ with density α, H_{Φ} is $(k, \alpha, \varepsilon_1, \varepsilon_2, \eta, \rho, p_1, p_2)$ -nice.

The proof of Lemma 3.13 is deferred to Appendix A. In fact, the same result also holds for random k-uniform hypergraphs. The following lemma states that if the incidence hypergraph of a random k-SAT formula satisfies some structural property with high probability, then a random hypergraph also satisfies the same property with high probability.

Lemma 3.14. Suppose $k \ge 3$ and α are constants. Let \mathbb{P} be a property for hypergraphs (i.e., \mathcal{P} is a subset of hypergraphs). If the probability of H_{Φ} belonging to \mathbb{P} is 1 - o(1/n), over the choice of random k-SAT formulas $\Phi \sim \Phi(k, n, \lfloor \alpha n \rfloor)$ with density α , then a random k-uniform hypergraph $H \sim H(k, n, \lfloor \alpha n \rfloor)$ with density α belongs to \mathbb{P} with probability 1 - o(1/n) as well.

Proof. Let $m = \lfloor \alpha n \rfloor$, and $\mathcal{E}_{k,n,m}$ be the event that the incidence hypergraphs H_{Φ} is a k-uniform hypergraph with *n* vertices and *m* distinct hyperedges. It is easy to see that

$$\mathbf{Pr}_{\Phi \sim \Phi(k,n,m)}[H_{\Phi} = H \mid \mathcal{E}_{k,n,m}] = \mathbf{Pr}_{H(k,n,m)}[H]$$

namely, the distribution of the incidence hypergraph H_{Φ} conditional on it being *k*-uniform and having *m* distinct hyperedges is the uniform distribution H(k, n, m).

Note that if *n* is sufficiently large, we obtain that

$$\begin{aligned} \mathbf{Pr}_{\Phi \sim \Phi(k,n,m)} \left[H_{\Phi} \text{ is } k \text{-uniform} \right] &= \left(\frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-k+1}{n} \right)^m \\ &\geq \left(1 - \frac{k-1}{n} \right)^{(k-1)m} \quad \geq \mathrm{e}^{-k^2 \alpha} \,, \end{aligned}$$

and

$$\mathbf{Pr}_{\Phi \sim \Phi(k,n,m)}[H_{\Phi} \text{ has two identical hyperedges}] \leq {\binom{m}{2}} \frac{k!}{n^k} \leq \frac{k!\alpha^2}{2n^k},$$

which further implies that

$$\mathbf{Pr}_{\Phi \sim \Phi(k,n,m)}[\mathcal{E}_{k,n,m}] \ge e^{-k^2\alpha} - \frac{k!\alpha^2}{2n^k} \ge \frac{1}{2}e^{-k^2\alpha}$$

for sufficiently large n. Thus, it follows that

$$\begin{aligned} & \mathbf{Pr}_{\Phi \sim \Phi(k,n,m)} [H_{\Phi} \notin \mathbb{P}] \\ & \geq \mathbf{Pr}_{\Phi \sim \Phi(k,n,m)} [H_{\Phi} \notin \mathbb{P} \mid \mathcal{E}_{k,n,m}] \cdot \mathbf{Pr}_{\Phi \sim \Phi(k,n,m)} [\mathcal{E}_{k,n,m}] \\ & \geq \frac{1}{2} e^{-k^{2} \alpha} \cdot \mathbf{Pr}_{H \sim H(k,n,m)} [H \notin \mathbb{P}] . \end{aligned}$$

Since $\operatorname{Pr}_{\Phi \sim \Phi(k,n,m)}[H_{\Phi} \notin \mathbb{P}] = o(1/n)$, we conclude that $\operatorname{Pr}_{H \sim H(k,n,m)}[H \notin \mathbb{P}] = o(1/n)$.

As a corollary, we have the following lemma immediately.

Lemma 3.15. For any fixed parameters $k, \alpha, \eta, \rho, \varepsilon_1, p_1, \varepsilon_2, p_2$ where $\eta k \ge 4$, $e(\rho k \alpha)^{\eta} \le 1$, $\varepsilon_1 = 2\eta$, $6k^5 \le p_1 \le e^{k-2}\alpha$, $\varepsilon_2 = 12k^5/((1-\eta)\eta p_1)$ and $p_2 = ek^2$, with probability 1 - o(1/n) over the choice of all k-uniform hypergraphs with n vertices and density α , a random k-uniform hypergraph H is $(k, \alpha, \varepsilon_1, \varepsilon_2, \eta, \rho, p_1, p_2)$ -nice.

We conclude this section by noting that Theorem 3.3 is simply a combination of Lemmas 3.13 and 3.15. Here, we do not need the lower bound on k in the conditions of Theorems 1.2 and 1.4 as the case of small k can be handled by taking a sufficiently large c.

4. Recursive coupling of random CSPs

In this section, we establish the following theorem regarding the decay of correlations for CSP instances that satisfy the structural properties introduced in the previous section. We note that Theorem 1.7 from introduction is an immediate consequence of Theorems 3.3 and 4.1.

Theorem 4.1. Let $\Phi = (V, [q], C)$ be an satisfiable atomic CSP formula satisfying Condition 3.1. Let $c_0 \in C$ be an arbitrary constraint. There exists a coupling (X, Y) of μ_C and $\mu_{C \setminus \{c_0\}}$ such that for any integer $\log n \leq M \leq \rho m$, it holds that

$$\Pr\left[d_{\operatorname{Ham}}(X,Y) \ge k \cdot M\right] \le 2^{-M}$$

Theorem 4.1 asserts that for CSP instances satisfying the specified structural properties, the discrepancy between uniform satisfying assignments induced by any particular constraint decays exponentially.

4.1. The coupling procedure. The coupling in Theorem 4.1 is the constraint-wise recursive coupling introduced in [WY24]. This coupling procedure, denoted as Couple($\mathcal{E}, \mathcal{F}, \sigma, \tau$), is formally described in Algorithm 2. The procedure takes as inputs:

- a pair of collections of pinned atomic constraints $\mathcal{E}, \mathcal{F} \subseteq C^*$, corresponding to two formulas;
- a pair of partial assignments $\sigma, \tau \in [q]^{\Lambda}$, specified on the same subset $\Lambda \subseteq V$ of variables.

It is assumed that both pinned formulas \mathcal{E}^{σ} and \mathcal{F}^{τ} are satisfiable. The objective of the procedure is to generate a pair of random assignments $(X, Y) \in [q]^V \times [q]^V$, such that marginally $X \sim \mu_{\mathcal{E}}^{\sigma}$ and $Y \sim \mu_{\mathcal{F}}^{\tau}$, while minimizing the discrepancy between *X* and *Y*.

The validity of this coupling is ensured by the following proposition. A similar correctness result was proven under a stronger local lemma condition in [WY24, Lemma 3.3].

Proposition 4.2. Assume that the atomic CSP formula $\Phi = (V, [q], C)$ is satisfiable. For any constraint $c_0 \in C$, the procedure Couple $(C \setminus \{c_0\}, C, \emptyset, \emptyset)$ terminates with probability 1 and returns a pair of random assignments $(X, Y) \in [q]^V \times [q]^V$ such that marginally $X \sim \mu_C \setminus \{c_0\}$ and $Y \sim \mu_C$.

Proof sketch. The proof follows the same inductive framework as the proof of [WY24, Lemma 3.3]. We provide a brief outline here.

First, by applying structural induction in the top-down order of recursion, one can verify the following induction hypothesis for each recursive call $Couple(\mathcal{E}, \mathcal{F}, \sigma, \tau)$:

$$\Lambda(\sigma) = \Lambda(\tau), \quad \mathcal{P}[\mathcal{E} \wedge \sigma] > 0, \quad \text{and} \quad \mathcal{P}[\mathcal{F} \wedge \tau] > 0.$$

This induction holds as long as the initial formula with constraints C is satisfiable, and it ensures that the coupling procedure remains well-defined throughout the recursion.

Algorithm 2: Couple($\mathcal{E}, \mathcal{F}, \sigma, \tau$) [WY24]

Instance: an atomic CSP formula $\Phi = (V, [q], C)$; :two subsets of pinned formulas $\mathcal{E}, \mathcal{F} \subseteq C^*$, and two partial assignments $\sigma, \tau \in [q]^{\Lambda}$ Input specified on the same subset $\Lambda \subseteq V$ of variables; **Output** : a pair of assignments $(X, Y) \in [q]^V \times [q]^V$; 1 if $\mathcal{E}^{\sigma} = \mathcal{F}^{\tau}$ then let (X, Y) be drawn according to the coupling of $\mu_{\mathcal{E}}^{\sigma}$ and $\mu_{\mathcal{F}}^{\tau}$ that always satisfies 2 $X_{V\setminus\Lambda}=Y_{V\setminus\Lambda};$ return (X, Y); 3 4 if $\mathcal{F}^{\tau} \not\subseteq \mathcal{E}^{\sigma}$ then choose the smallest $c \in \mathcal{F}^{\tau} \setminus \mathcal{E}^{\sigma}$; 5 with probability $\mu_{\mathcal{E}}^{\sigma}(c)$ do | return Couple ($\mathcal{E} \cup \{c\}, \mathcal{F}, \sigma, \tau$); 6 7 8 else let $\pi = \texttt{False}(c)$ and draw a random $\rho \sim \mu_{\mathcal{F}, \texttt{vbl}(c)}^{\tau}$; 9 **return** Couple $(\mathcal{E}, \mathcal{F}, \sigma \land \pi, \tau \land \rho)$; 10 else 11 choose the smallest $c \in \mathcal{E}^{\sigma} \setminus \mathcal{F}^{\tau}$; 12 with probability $\mu_{\mathcal{F}}^{\tau}(c)$ do 13 **return** Couple $(\mathcal{E}, \mathcal{F} \cup \{c\}, \sigma, \tau)$; 14 else 15 draw a random $\pi \sim \mu_{\mathcal{E}, \text{vbl}(c)}^{\sigma}$ and let $\rho = \text{False}(c)$; 16 **return** Couple $(\mathcal{E}, \mathcal{F}, \sigma \land \pi, \tau \land \rho)$; 17

Next, observe that in each recursive step, either the size of the symmetric difference $\mathcal{E}^{\sigma} \triangle \mathcal{F}^{\tau}$ decreases by one, or the number of unassigned variables in σ and τ is reduced by at least one. Since Algorithm 2 terminates when $\mathcal{E}^{\sigma} = \mathcal{F}^{\tau}$, the procedure Couple($\mathcal{E}, \mathcal{F}, \sigma, \tau$) eventually terminates due to the finiteness of both the number of constraints and the number of variables.

Finally, by applying structural induction in the bottom-up order of recursion, we can verify the following induction hypothesis to ensure the correctness of the coupling:

Couple($\mathcal{E}, \mathcal{F}, \sigma, \tau$) returns an (*X*, *Y*) such that marginally $X \sim \mu_{\mathcal{E}}^{\sigma}$ and $Y \sim \mu_{\mathcal{F}}^{\tau}$.

This induction follows the same steps as the one given in the proof of [WY24, Lemma 3.3]. The correctness holds as long as the coupling procedure is well-defined and terminates, which we have already established. \Box

4.2. The witness tree and witness assignment. The novelty of Theorem 4.1 lies in an improved analysis of the coupling procedure (Algorithm 2). This new analysis of the coupling establishes an exponential decay of correlation by leveraging new structural properties of the underlying hypergraph for the CSP formula. The main slackness in previous analysis of the coupling [WY24] arises from the use of a combinatorial structure called a 2-tree, originally introduced in [Alo91], serving as a witness for a large discrepancy between satisfying assignments. To achieve an improved analysis, we employ a witness tree structure, constructed similarly to the witness tree used in the analysis of the Moser-Tardos algorithm [MT10]. This replaces the 2-tree with a new certificate for the discrepancy in the coupling. This use of the Moser-Tardos witness tree is crucial for approaching the satisfiability threshold.

We begin with a formal definition of the execution log for Algorithm 2.

Definition 4.3 (execution log). Given a run of Algorithm 2 from Couple($C \setminus \{c_0\}, C, \emptyset, \emptyset$), the *execution* log $L = L(C, c_0) = (c_1, c_2, ..., c_\ell)$ is a random sequence of (unpinned) constraints from *C*, constructed as follows:

• initialize *L* as the empty list;

• whenever $\text{Couple}(\mathcal{E}, \mathcal{F}, \sigma \land \pi, \tau \land \rho)$ is recursively called at Line 10 or Line 17, append the original unpinned constraint $c^{O} \in C$ to the end of *L*.

For any sequence $(c_1, c_2, ..., c_\ell)$ of (unpinned) constraints from *C*, we say that $(c_1, c_2, ..., c_\ell)$ is a *proper execution log* if it appears in the support of $L = L(C, c_0)$, i.e.,

Pr
$$[L = (c_1, c_2, \dots, c_\ell)] > 0.$$

Recall the definition of the incidence hypergraph in Definition 2.2. The following notion of a witness tree is inspired by the witness tree introduced for analyzing the Moser-Tardos algorithm [MT10]. A key distinction in our context is that in the execution log constructed as in Definition 4.3, a constraint can never appear twice. Thus, the witness tree here is a subgraph of G_{Φ} .

Definition 4.4 (witness tree). A *witness tree* T is a finite rooted tree with vertex set $V(T) \subseteq C$ such that T is a subgraph of G_{Φ} .

Given a witness tree *T* and a constraint $c \in C$, define $T \oplus c$ as the witness tree *T'* obtained as follows:

- if $T = \emptyset$, then T' is the rooted tree containing a single vertex c;
- otherwise, if $\exists c' \in V(T)$ such that $vbl(c) \cap vbl(c') \neq \emptyset$, then T' is obtained by adding c as a child of the deepest such c' in T, breaking ties by choosing the lexicographically smallest c' in case multiple c's with the same largest depth exist;
- otherwise, T' = T.

Given a proper execution log $L = (c_1, c_2, ..., c_\ell)$, the *witness tree of the execution log L*, denoted by T = T(L), is defined as follows: Let $T_i = T_{i-1} \oplus c_i$ for $1 \le i \le \ell$ and initialize $T_0 = \emptyset$. Thus, $T(L) = T_\ell$.

The following proposition states several basic properties of the witness tree. The proof is straightforward and is therefore omitted.

Proposition 4.5. Let *L* be a proper execution log, and let T = T(L) be the rooted tree constructed by Definition 4.4. For each $c \in T$, denote the depth of *c* in *T* as dep(*c*). The following properties hold:

- (1) T is a witness tree, i.e., T is a rooted tree and is a subgraph of G_{Φ} .
- (2) |V(T)| = |L|, meaning that every constraint in L is included as a vertex in T.
- (3) For any distinct $c, c' \in V(T)$ with dep(c) = dep(c'), it holds that $vbl(c) \cap vbl(c') = \emptyset$.
- (4) For any $c, c' \in V(T)$, if $vbl(c) \cap vbl(c') \neq \emptyset$ and c' appears later than c in the execution log L, then dep(c') > dep(c).

Besides the witness tree, another main source of randomness in the coupling procedure of Algorithm 2 comes from the non-violating partial assignments drawn in Lines 9 and 16 of Algorithm 2. This is formalized by the following notion of witness assignment.

Definition 4.6 (witness assignment). Given a run of Algorithm 2 from $Couple(C \setminus \{c_0\}, C, \emptyset, \emptyset)$, the *witness assignment* is a random partial assignment ς constructed as follows:

- initially, $\varsigma = \emptyset$, the empty assignment;
- whenever a recursive call $Couple(\mathcal{E}, \mathcal{F}, \sigma \land \pi, \tau \land \rho)$ is made in Line 10 of Algorithm 2, with $\pi = False(c)$ being the unique violating assignment of constraint *c*, update $\varsigma \leftarrow \varsigma \land \rho$;
- whenever a recursive call $Couple(\mathcal{E}, \mathcal{F}, \sigma \land \pi, \tau \land \rho)$ is made in Line 17 of Algorithm 2, with $\rho = False(c)$ being the unique violating assignment of constraint *c*, update $\varsigma \leftarrow \varsigma \land \pi$.

For any witness tree *T*, we define

$$\operatorname{vbl}(T) \triangleq \bigcup_{c \in V(T)} \operatorname{vbl}(c),$$

which represents the set of all variables involved in the constraints that appear as vertices in *T*. Based on Proposition 4.5 and definition 4.6, it follows that the witness assignment ς is a partial assignment specified over the variables in vbl(*T*), i.e. $\varsigma \in [q]^{\text{vbl}(T)}$, where T = T(L) is the witness tree of the execution log *L* that belongs to a run of Algorithm 2 from Couple($C \setminus \{c_0\}, C, \emptyset, \emptyset$).

The witness tree, together with the witness assignment, fully determines the random choices made by Algorithm 2, except for the optimal coupling step at the end of the recursion.

Lemma 4.7. Let T be any tree in G_{Φ} rooted at c_0 , and let $\varsigma \in [q]^{\text{vbl}(T)}$ be a partial assignment. Given the witness tree of the execution log being T and the witness assignment being ς , a run of Algorithm 2 from Couple($C \setminus \{c_0\}, C, \emptyset, \emptyset$) is fully determined, except for the random choices made in Line 2.

Proof. For each $c \in V(T)$, define var(c) as the set of variables within vbl(c), excluding all variables within vbl(c') for any $c' \in V(T)$ such that dep(c') < dep(c) in T and $vbl(c') \cap vbl(c) \neq \emptyset$. Note that var(c) is determined by the witness tree *T*.

Then, the random choices in Line 6 (and Line 13) can be determined by checking whether $c^{O} \in V(T)$ and $vbl(c) = var(c^{O})$ for the pinned constraint $c \in C^*$ picked in Line 5 (and Line 12). Recall that c^{O} denotes the original unpinned constraint from C corresponding to c.

The random assignments generated in Lines 10 and 17 are recorded by the witness assignment ς , and thus can be fully recovered from ς .

Therefore, a run of Algorithm 2 from $Couple(C \setminus \{c_0\}, C, \emptyset, \emptyset)$ can be deterministically simulated, except for the random assignments generated in Line 2. П

4.3. Analysis of the coupling. In this subsection, we utilize the witness tree and witness assignment defined previously to prove Theorem 4.1.

The following definition describes a random process to simulate Algorithm 2 using both the witness tree and witness assignment, employing the principle of deferred decisions.

Definition 4.8 (M-truncated process for simulating Algorithm 2 with explicitly identified randomness). Let $\mathfrak{X} \sim \mu_{C \setminus \{c_0\}}$ and $\mathfrak{Y} \sim \mu_C$ be drawn independently beforehand. Define the random process

$$\mathbf{P}^{cp} = \{ (\mathcal{E}_t, \mathcal{F}_t, \sigma_t, \tau_t, T_t, \varsigma_t) \}_{t \ge 0}$$

starting from the initial state $(\mathcal{E}_0, \mathcal{F}_0, \sigma_0, \tau_0, T_t, \varsigma_t) = (C \setminus \{c_0\}, C, \emptyset, \emptyset, \emptyset)$ as follows:

- (1) If $\mathcal{E}_t^{\sigma_t} = \mathcal{F}_t^{\tau_t}$ or $|V(T_t)| = M$, the process stops, and $(\mathcal{E}_t, \mathcal{F}_t, \sigma_t, \tau_t, T_t, \varsigma_t)$ is the outcome. (2) Otherwise, if $\mathcal{F}_t^{\tau_t} \nsubseteq \mathcal{E}_t^{\sigma_t}$, let *c* be the smallest pinned constraint in $\mathcal{F}_t^{\tau_t} \setminus \mathcal{E}_t^{\sigma_t}$. We then set $(\mathcal{E}_{t+1}, \mathcal{F}_{t+1}, \sigma_{t+1}, \tau_{t+1}, T_{t+1}, \varsigma_{t+1})$ as

$$\begin{cases} (\mathcal{E}_t \cup \{c\}, \mathcal{F}_t, \sigma_t, \tau_t, T_t, \varsigma_t) & \text{if } c \text{ is satisfied by } \mathfrak{X}; \\ (\mathcal{E}_t, \mathcal{F}_t, \sigma_t \wedge \mathfrak{X}_{\text{vbl}(c)}, \tau_t \wedge \mathfrak{Y}_{\text{vbl}(c)}, T_t \oplus c^O, \varsigma_t \wedge \mathfrak{Y}_{\text{vbl}(c)}) & \text{otherwise.} \end{cases}$$

Here, $c^{O} \in C$ denotes the original unpinned version of c, and $T_t \oplus c^{O}$ is given in Definition 4.4. (3) Otherwise, if $\mathcal{F}_t^{\tau_t} \subseteq \mathcal{E}_t^{\sigma_t}$, let c be the smallest pinned constraint in $\mathcal{E}_t^{\sigma_t} \setminus \mathcal{F}_t^{\tau_t}$. We then set $(\mathcal{E}_{t+1}, \mathcal{F}_{t+1}, \sigma_{t+1}, \tau_{t+1}, T_{t+1}, \varsigma_{t+1})$ as

;

$$\begin{cases} (\mathcal{E}_t, \mathcal{F}_t \cup \{c\}, \sigma_t, \tau_t, T_t, \varsigma_t) & \text{if } c \text{ is satisfied by } \mathfrak{Y} \\ (\mathcal{E}_t, \mathcal{F}_t, \sigma_t \wedge \mathfrak{X}_{\text{vbl}(c)}, \tau_t \wedge \mathfrak{Y}_{\text{vbl}(c)}, T_t \oplus c^O, \varsigma_t \wedge \mathfrak{X}_{\text{vbl}(c)}) & \text{otherwise.} \end{cases}$$

Let μ^{cp} denote the distribution of the outcome $(\mathcal{E}^{cp}, \mathcal{F}^{cp}, \sigma^{cp}, \tau^{cp}, \tau^{cp}, \varsigma^{cp})$ of this process, and let $\mathcal{L}^{cp} = \operatorname{supp}(\mu^{cp})$ be its support. Let \mathcal{V}^{cp} denote the set of all possible $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)$ such that

$$\Pr\left[\left(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma\right)\in P^{\mathrm{cp}}\right]>0.$$

Remark 4.9 (distinctions between Algorithm 2 and Definition 4.8). We note two main distinctions between Algorithm 2 and Definition 4.8, specifically regarding truncation and the explicitly identified randomness using the principle of deferred decisions. Both aspects are beneficial for our analyses of the coupling and algorithmic applications.

- In Definition 4.8, the process is truncated once the size of the witness tree reaches the threshold *M*, whereas Algorithm 2 does not incorporate such truncation.
- In Definition 4.8, all randomness utilized by the process P^{cp} is identified with the two pregenerated random assignments $\mathfrak{X} \sim \mu_{C \setminus c_0}$ and $\mathfrak{Y} \sim \mu_C$. In contrast, Algorithm 2 generates random choices at the moment they are needed. However, by the principle of deferred decisions, Definition 4.8 still faithfully simulates Algorithm 2 (truncated when the size of the witness tree reaches M). Thus, properties we proved for Algorithm 2 (specifically Proposition 4.5 and Lemma 4.7) also hold for the random process constructed in Definition 4.8.

The above observation is formalized by the following lemma that upper bounds the correlation decay of the coupling by the probability of the truncation of the random process P^{cp} . The proof of this lemma is similar to that of [WY24, Lemma 3.9], and is included for completeness.

Lemma 4.10. Assume Φ is satisfiable. Let (X, Y) be the output of $\text{Couple}(C \setminus \{c_0\}, C, \emptyset, \emptyset)$, and let $(\mathcal{E}^{cp}, \mathcal{F}^{cp}, \sigma^{cp}, \tau^{cp}, \tau^{cp}, \varsigma^{cp}) \sim \mu^{cp}$ be the outcome of the process P^{cp} . Then, we have

$$\Pr\left[d_{\operatorname{Ham}}(X,Y) \ge k \cdot M\right] \le \Pr\left[|V(T^{\operatorname{cp}})| = M\right].$$

Proof. By the construction of witness tree in Definition 4.4 and Proposition 4.5, each time a recursive call is made to Couple($\mathcal{E}, \mathcal{F}, \sigma \land \pi, \tau \land \rho$) at Line 10 or Line 17 of Algorithm 2, at most *k* variables are assigned into σ and τ , and the size of the current witness tree T = T(L) increases by one. Furthermore, by Line 3 of Algorithm 2, the Hamming distance between *X* and *Y* is upper bounded by the total number of variables assigned in σ and τ during the recursion Couple($C \setminus \{c_0\}, C, \emptyset, \emptyset$). Hence, we have

$$d_{\operatorname{Ham}}(X,Y) \ge k \cdot M \implies |V(T)| \ge M.$$

We now argue that the witness tree T = T(L), constructed from the execution log L of a run of Algorithm 2 from Couple($C \setminus \{c_0\}, C, \emptyset, \emptyset$), can be coupled with the witness tree T^{cp} constructed during the process P^{cp} described in Definition 4.8. Specifically, whenever a recursive call to Couple($\mathcal{E}, \mathcal{F}, \sigma, \tau$) is made in Algorithm 2, with the current witness tree being T and witness assignment ς , the process P^{cp} moves to a state ($\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma$). Under this coupling, the lemma follows immediately.

It is important to note that this coupling between the two processes, Algorithm 2 and P^{cp} , does not hold trivially, because the randomness used in the construction of P^{cp} is explicitly identified to the random satisfying assignments:

$$\mathfrak{X} \sim \mu_{C \setminus \{c_0\}}$$
 and $\mathfrak{Y} \sim \mu_C$.

Nevertheless, this perfect coupling between Algorithm 2 and P^{cp} can be verified by structural induction, proceeding in the top-down order of recursion, with a strengthened hypothesis: conditioning on the process P^{cp} being at the state ($\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma$), Algorithm 2 is at the same state, and it further holds:

(3)
$$\mathfrak{X} \sim \mu_{\mathcal{E}}^{\sigma}$$
 and $\mathfrak{Y} \sim \mu_{\mathcal{F}}^{\tau}$

With (3), the transition probabilities to the next state are identical in both processes. Therefore, the two processes are perfectly coupled. Additionally, it can be verified that (3) continues to hold for each possible branch of the process P^{cp} . This concludes the proof of the lemma.

The proof of Theorem 4.1 now reduces to establishing an upper bound of 2^{-M} on the truncation probability $\Pr[|V(T^{cp})| = M]$. A crucial step in this process is the following technical lemma, which assumes Condition 3.1 and will also play a key role in the algorithmic implication of the coupling later.

Lemma 4.11. Assume Φ is satisfiable. Let $\mathfrak{X} \sim \mu_{C \setminus \{c_0\}}$ and $\mathfrak{Y} \sim \mu_C$. For any subset of variables $S \subseteq V$,

$$\begin{aligned} \forall X \in [q]^S, \qquad \mu_{C \setminus \{c_0\}}(X) = \mathbf{Pr} \left[\mathfrak{X}_S = X\right] &\leq q^{-|S \cap V_{\text{good}}|} \cdot (1 - eq^{-(1-\varepsilon_1)k})^{-|S|p_1\alpha}; \\ \forall Y \in [q]^S, \qquad \mu_C(Y) = \mathbf{Pr} \left[\mathfrak{Y}_S = Y\right] &\leq q^{-|S \cap V_{\text{good}}|} \cdot (1 - eq^{-(1-\varepsilon_1)k})^{-|S|p_1\alpha}. \end{aligned}$$

Proof. We only prove the first inequality, and the second one follows similarly. For any $\tau \in [q]^{V_{\text{bad}}}$ with $\Pr[\mathfrak{X}_{V_{\text{bad}}} = \tau] > 0$, the following bound holds:

(4)
$$\mathbf{Pr}\left[\mathfrak{X}_{\mathrm{vbl}(T)\cap V_{\mathrm{good}}} = X_{\mathrm{vbl}(T)\cap V_{\mathrm{good}}} \mid \mathfrak{X}_{V_{\mathrm{bad}}} = \tau\right] \le q^{-|\mathrm{vbl}(T)\cap V_{\mathrm{good}}|} \cdot (1 - \mathrm{e}q^{-(1-\varepsilon_1)k})^{-|\mathrm{vbl}(T)|p_1\alpha}.$$

This is because, given the pinning $\mathfrak{X}_{V_{\text{bad}}} = \tau$, the resulting pinned formula has each variable with a degree of at most p_1d (by Fact 3.8) and each constraint containing at least $(1 - \varepsilon_1)k$ variables (by Fact 3.9). Applying Theorem 2.4 by setting the parameter $x(c) = eq^{-(1-\varepsilon_1)k}$ for each constraint *c* in the pinned formula, we derive the bound in (4).

Now, we can bound the probability of $\mathfrak{X}_{vbl(T)} = X$ as follows:

$$\mathbf{Pr}\left[\mathfrak{X}_{\mathrm{vbl}(T)} = X\right] = \sum_{\tau \in [q]^{V_{\mathrm{bad}}}} \mathbf{Pr}\left[\mathfrak{X}_{V_{\mathrm{bad}}} = \tau\right] \cdot \mathbf{Pr}\left[\mathfrak{X}_{\mathrm{vbl}(T) \cap V_{\mathrm{good}}} = X_{\mathrm{vbl}(T) \cap V_{\mathrm{good}}} \mid \mathfrak{X}_{V_{\mathrm{bad}}} = \tau\right]$$

$$(\mathrm{by}(4)) \leq q^{-|\mathrm{vbl}(T) \cap V_{\mathrm{good}}|} \cdot (1 - \mathrm{e}q^{-(1-\varepsilon_{1})k})^{-|\mathrm{vbl}(T)|} p_{1}\alpha.$$

We are now ready to prove Theorem 4.1, the main theorem of this section.

Proof of Theorem 4.1. Consider the outcome $(\mathcal{E}^{cp}, \mathcal{F}^{cp}, \sigma^{cp}, \tau^{cp}, \varsigma^{cp}) \sim \mu^{cp}$ of the random process P^{cp} in Definition 4.8. We first show that for any rooted tree T in G_{Φ} and any $\varsigma \in [q]^{\operatorname{vbl}(T)}$, we have

(5)
$$\mathbf{Pr}\left[T^{\mathrm{cp}} = T \wedge \varsigma^{\mathrm{cp}} = \varsigma\right] \le q^{-2|\mathrm{vbl}(T) \cap V_{\mathrm{good}}|} \cdot \left(1 - \mathrm{e}q^{-(1-\varepsilon_1)k}\right)^{-2|\mathrm{vbl}(T)|p_1\alpha}$$

Recall the pre-generated $\mathfrak{X} \sim \mu_{C \setminus \{c_0\}}$ and $\mathfrak{Y} \sim \mu_C$, used in the construction of the process P^{cp} and ς^{cp} . By Lemma 4.7, there are two partial assignments $X, Y \in [q]^{vbl(T)}$ determined by T and ς such that

(6)
$$T^{\rm cp} = T \wedge \varsigma^{\rm cp} = \varsigma \implies \mathfrak{X}_{{\rm vbl}(T)} = X \wedge \mathfrak{Y}_{{\rm vbl}(T)} = Y.$$

Since \mathfrak{X} and \mathfrak{Y} are independent, the inequality in (5) follows from (6) and Lemma 4.11.

We proceed to prove Theorem 4.1. According to Definition 4.8, all nonempty witness trees T^{cp} must be rooted at c_0 . Let $\mathbb{T}_M^{c_0}$ denote the set of witness trees of size M rooted at c_0 . Then, we have

$$\Pr\left[d_{\operatorname{Ham}}(X,Y) \ge k \cdot M\right]$$
(by Lemma 4.10)
$$\leq \Pr\left[|V(T^{\operatorname{cp}})| = M\right]$$

$$\leq \sum_{T \in \mathbb{T}_{M}^{c_{0}}} \sum_{\varsigma \in [q]^{\operatorname{vbl}(T)}} \Pr\left[T^{\operatorname{cp}} = T \land \varsigma^{\operatorname{cp}} = \varsigma\right]$$
(by (5))
$$\leq \sum_{T \in \mathbb{T}_{M}^{c_{0}}} \sum_{\varsigma \in [q]^{\operatorname{vbl}(T)}} q^{-2|\operatorname{vbl}(T) \cap V_{\operatorname{good}}|} \cdot \left(1 - \operatorname{e} q^{-(1-\varepsilon_{1})k}\right)^{-2|\operatorname{vbl}(T)|p_{1}\alpha}$$

$$\leq \sum_{T \in \mathbb{T}_{M}^{c_{0}}} q^{|\operatorname{vbl}(T)|} \cdot q^{-2|\operatorname{vbl}(T) \cap V_{\operatorname{good}}|} \cdot \left(1 - \operatorname{e} q^{-(1-\varepsilon_{1})k}\right)^{-2|\operatorname{vbl}(T)|p_{1}\alpha}.$$

Note that for $T \in \mathbb{T}_{M}^{c_{0}}$, the size of vbl(*T*) can be easily upper bounded by *kM*. We then lower bound $|vbl(T) \cap V_{good}|$ as follows:

$$\begin{aligned} \left| \operatorname{vbl}(T) \cap V_{\text{good}} \right| &= \left| \operatorname{vbl}(T \cap \mathcal{E}_{\text{good}}) \cap V_{\text{good}} \right| \\ (\text{by Fact 3.9}) &\geq \left| \operatorname{vbl}(T \cap \mathcal{E}_{\text{good}}) \right| - \varepsilon_1 kM \\ (\text{by Properties 3.5 and 3.11}) &\geq (1 - \eta)(1 - \varepsilon_2) kM - \varepsilon_1 kM. \end{aligned}$$

Therefore, we have

(7)

$$\Pr\left[d_{\operatorname{Ham}}(X,Y) \ge k \cdot M\right]$$

$$\leq \sum_{T \in \mathbb{T}_{M}^{c_{0}}} q^{-(2(1-\eta)(1-\varepsilon_{2})-(1+2\varepsilon_{1}))kM} \cdot \left(1 - eq^{-(1-\varepsilon_{1})k}\right)^{-2kMp_{1}\alpha}$$
(by Property 3.6)
$$\leq n^{3} \cdot (p_{2}\alpha)^{M} \cdot q^{-(2(1-\eta)(1-\varepsilon_{2})-(1+2\varepsilon_{1}))kM} \cdot (1 - eq^{-(1-\varepsilon_{1})k})^{-2kMp_{1}\alpha}$$

$$\leq \left(2^{3\log n/M} \cdot p_{2}\alpha \cdot q^{-(2(1-\eta)(1-\varepsilon_{2})-(1+2\varepsilon_{1}))k} \cdot \left(1 - eq^{-(1-\varepsilon_{1})k}\right)^{-2kp_{1}\alpha}\right)^{M}.$$

We upper bound each term in the bracket separately.

- Since $M \ge \log n$, it holds $2^{3\log n/M} \le 8$;
- Since $p_2 = ek^2$, it holds $p_2\alpha = ek^2\alpha$;
- Since $\eta = \Theta(1/k)$, $\varepsilon_1 = \Theta(1/k)$ and $\varepsilon_2 = \Theta(1/k)$, we deduce that

$$2(1-\eta)(1-\varepsilon_2)-(1+2\varepsilon_1)\geq 1-2(\eta+\varepsilon_1+\varepsilon_2)=1-\Theta\left(\frac{1}{k}\right),$$

and hence it holds

$$q^{-(2(1-\eta)(1-\varepsilon_2)-(1+2\varepsilon_1))k} \le q^{-k+O(1)};$$

• Since $\varepsilon_1 = \Theta(1/k)$ and $p_1 = \Theta(k^7)$, for k sufficiently large it holds

$$\left(1 - eq^{-(1-\varepsilon_1)k}\right)^{-2kp_1\alpha} = \exp\left(\Theta\left(eq^{-(1-\varepsilon_1)k} \cdot 2kp_1\alpha\right)\right)$$
$$= \exp\left(\Theta\left(q^{-k+O(1)}k^8\alpha\right)\right)$$
$$= O(1),$$

assuming $\alpha \leq \frac{q^k}{(qk)^c}$ for a sufficiently large universal constant $c \geq 8$. Combining everything above, we conclude with

$$\Pr\left[d_{\operatorname{Ham}}(X,Y) \ge k \cdot M\right] \le \left(8ek^2\alpha \cdot q^{-k+O(1)} \cdot O(1)\right)^M \le 2^{-M},$$

where we assume $\alpha \leq \frac{q^k}{(qk)^c}$ for some sufficiently large universal constant c > 0.

4.4. **Proofs of replica symmetry and non-reconstruction.** In this subsection, we leverage the coupling analysis from previous sections to explore the correlation decay properties for random CSP formulas. Specifically, we will establish the properties of replica symmetry (Theorem 1.9) and non-reconstruction (Theorem 1.11) for random CSPs at the considered densities. Additionally, we will examine a connectivity property of the solution space, known as the *looseness* property (which will be formally defined later in Definition 4.12).

It is important to note that while these theorems are stated in the context of random *k*-SAT, our proofs are applicable to a broader class of random CSPs, including random hypergraph colorings.

Proof of Theorems 1.9 and 1.11. Recall the decay of correlation property established in Theorem 4.1, which implies both replica symmetry and non-reconstruction.

We apply the procedure $\text{Couple}(\mathcal{E}, \mathcal{F}, \sigma, \tau)$ described in Algorithm 2 to construct a coupling between two instances obtained from the same set of constraints *C*, differing only in the pinning on one variable. Specifically, we consider the coupling procedure $\text{Couple}(\mathcal{E}, \mathcal{F}, \sigma, \tau)$ with the following initial states:

- The initial set of constraints are $\mathcal{E} = \mathcal{F} = C$;
- The initial assignments σ , τ are both specified on just v, with $\sigma(v) = x_1$ and $\tau(v) = x_2$.

The key proofs for the coupling procedure in Section 4 apply to this setting. In particular, the correctness follows directly from the proof of Proposition 4.2. Thus, $\text{Couple}(\mathcal{E}, \mathcal{F}, \sigma, \tau)$ returns (X, Y) such that $X \sim \mu_C^{\sigma}$ and $Y \sim \mu_C^{\tau}$. Since all pinned constraints in the initial discrepancy set $\mathcal{E}^{\sigma} \triangle \mathcal{F}^{\tau}$ must include v, they are connected in G_{Φ} . Hence, the proof for Proposition 4.5 holds in this case as well. As a result, following the same reasoning as in Theorem 4.1, we can establish that for the pair (X, Y) returned by this $\text{Couple}(\mathcal{E}, \mathcal{F}, \sigma, \tau)$, we have

$$\Pr\left[d_{\operatorname{Ham}}(X,Y) \ge k \cdot M\right] \le d \cdot 2^{-M}.$$

The extra factor of *d* in the probability bound comes from the need to bound the number of witness trees rooted at a constraint containing *v*, rather than at a single constraint c_0 . While Condition 3.1 technically holds with width k - 1 after pinning one variable, the proof of Theorem 4.1 is robust enough to proceed with this slightly weakened Condition 3.1, thereby implying (8).

Finally, we conclude the proof:

- Theorem 1.9 follows from the exponential decay of correlation established in (8);
- Theorem 1.11 also follows from (8), as with high probability, the discrepancy at v does not affect the assignment on $\bar{B}_H(v, r)$ when r is large enough.

Recent work has focused on exploring the solution space of random k-SAT through various notions of connectivity. One way to characterize connectivity of the solution space is looseness considered in [AC08, CGG⁺24], basically saying one can obtain another solution with some variable flipped without overall flipping too many variables.

Definition 4.12 (looseness). Let $\Phi = (V, C)$ be a SAT formula with |V| = n. A variable $v \in V$ is said to be *M*-loose with respect to a satisfying assignment $\sigma \in \Omega_{\Phi}$ if there exists another satisfying assignment $\tau \in \Omega_{\Phi}$ such that $\tau(v) \neq \sigma(v)$ and $d_{\text{Ham}}(\sigma, \tau) \leq M$.

For a random *k*-SAT formula $\Phi \sim \Phi(k, n, \lfloor \alpha n \rfloor)$, we say that Φ is *M*-loose if, with high probability over the pair (Φ, σ) , where $\sigma \sim \mu_{\Phi}$, all variables $v \in V$ are *M*-loose with respect to σ .

Looseness is conjectured to hold for random *k*-SAT up to the clustering threshold $\alpha_{\text{clust}} \approx 2^k (\ln k)/k$ [AC08, Conjecture 1]. Here, we prove looseness at the considered densities.

Theorem 4.13 (looseness of random *k*-SAT). Under the condition of Theorem 1.2, the random *k*-SAT formula $\Phi(k, n, \lfloor \alpha n \rfloor)$ is (poly(*k*) log *n*)-loose.

Proof. We will prove Theorem 4.13, by modifying the random process introduced in Definition 4.8 to find a local neighbor of a solution. The modified process proceeds as follows.

The input consists of an atomic CSP formula $\Phi = (V, [q], C)$, a solution $\sigma^{\text{in}} \in \Omega_{\Phi}$, and a variable $v \in V$. The process produces an output assignment $\sigma^{\text{out}} \in \Omega_{\Phi}$ as described below:

- (1) Choose an arbitrary value $x \in [q] \setminus \{\sigma^{in}(v)\}$.
- (2) Run the random process in Definition 4.8 with the following alterations:
 - The initial sets of constraints are *E*₀ = *F*₀ = *C*, and the initial partial assignments are both specified only at *v*, with *σ*₀(*v*) = *σ*ⁱⁿ(*v*) and *τ*₀(*v*) = *x*;
 - Set \mathfrak{X} as σ^{in} and $\mathfrak{Y} \sim \mu_C^{\tau_0}$. That is, $\mathfrak{Y} \in \Omega^C$ is chosen uniformly at random conditioned on $\mathfrak{Y}(v) = x$.
 - Let $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)$ be the outcome of the random process.
- (3) If the outcome (ε, F, σ, τ, T, ς) satisfy that |T| = M, then let σ^{out} ← σⁱⁿ. Otherwise, update σⁱⁿ by changing the values of the variables assigned in τ to τ, i.e. set σ^{out} ← τ ∧ σⁱⁿ_{V\Λ(τ)}.

We can make the following observations about this process:

- Similar to Algorithm 2 or Definition 4.8, this random process is an idealized algorithm, as it requires sampling from the non-trivial distribution $\mu_C^{\tau_0}$. The process is constructed to prove the looseness of random CSP formulas (Theorem 4.13).
- In Item 2, the sets of pinned constraints *ε^σ* and *F^τ* may have discrepancies at the constraints involving *v*. Since these constraints are connected in *G*_Φ, the proof for Proposition 4.5 hold in this setting as well.
- In Item 3, if the outcome $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)$ satisfies |T| < M, then by Definition 4.8, we must have $\mathcal{E}^{\sigma} = \mathcal{F}^{\tau}$. Since $\mathfrak{X} = \sigma^{\text{in}}$, the output assignment σ^{out} is a satisfying assignment with $\sigma^{\text{out}}(v) \neq \sigma^{\text{in}}(v)$ and $d_{\text{Ham}}(\sigma^{\text{in}}, \sigma^{\text{out}}) \leq kM$.

Finally, Theorem 4.13 follows from applying the same argument used in the proofs of Theorems 1.9 and 1.11, by invoking Theorem 4.1 to the setting where the initial instances differ in one variable rather than in one constraint. \Box

5. LINEAR PROGRAM FOR COUNTING AND SAMPLING

In this section, we introduce a linear programming approach that translates the coupling result from the previous section into algorithms for counting and sampling, and prove Theorem 3.2. Similar to the coupling, this linear program is adapted from the one in [WY24], with key modifications to accommodate the criticality of random instances.

5.1. **Marginal probabilities.** We introduce the marginal probabilities associated with the coupling procedure, which correspond to the variables of the linear program.

Consider an atomic CSP formula $\Phi = (V, [q], C)$, and let $c_0 \in C$ be an arbitrary constraint. Fix an integer $M \ge 1$. Recall the random process $P^{cp} = P_M^{cp} = \{(\mathcal{E}_t, \mathcal{F}_t, \sigma_t, \tau_t, T_t, \varsigma_t)\}_{t\ge 0}$ and its outcome distribution $\mu^{cp} = \mu_M^{cp}$, along with their supports \mathcal{V}^{cp} and \mathcal{L}^{cp} , as constructed in Definition 4.8.

We define a family of marginal probabilities induced by the random process P^{cp} .

Definition 5.1 (marginal probabilities). Let *X* and *Y* be generated as follows:

- draw $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \sim \mu^{cp}$;
- draw $X \sim \mu_{\mathcal{E}}^{\sigma}$, and similarly $Y \sim \mu_{\mathcal{F}}^{\tau}$.

For each $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{V}^{cp}$, we define the following pair of marginal probabilities:

$$p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{X} \triangleq \Pr\left[(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma) \in P^{\operatorname{cp}} \mid X = \mathbf{x}\right],$$
$$p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{Y} \triangleq \Pr\left[(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma) \in P^{\operatorname{cp}} \mid Y = \mathbf{y}\right].$$

where $\mathbf{x}, \mathbf{y} \in [q]^V$ are arbitrary assignments satisfying $\mathcal{E} \wedge \sigma$ and $\mathcal{F} \wedge \tau$, respectively.

Note that each marginal probability $p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^X$ is defined by conditioning on X being an arbitrary assignment $\mathbf{x} \in [q]^V$ that satisfies $\mathcal{E} \wedge \sigma$ (and similarly for each $p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^Y$). The well-definedness of these probabilities is ensured by the following proposition.

Proposition 5.2. Assume Φ is satisfiable. Fix any $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{V}^{cp}$. The following sets are nonempty:

$$\Omega^{\mathcal{E}\wedge\sigma} \triangleq \{\pi \in [q]^V \mid \pi \text{ satisfies } \mathcal{E}\wedge\sigma\},\$$
$$\Omega^{\mathcal{F}\wedge\tau} \triangleq \{\pi \in [q]^V \mid \pi \text{ satisfies } \mathcal{F}\wedge\tau\}.$$

Furthermore, for any $\mathbf{x}, \mathbf{x}' \in \Omega^{\mathcal{E} \wedge \sigma}$ and $\mathbf{y}, \mathbf{y}' \in \Omega^{\mathcal{F} \wedge \tau}$, it holds that

$$\mathbf{Pr} \left[(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\mathrm{cp}} \mid X = \mathbf{x} \right] = \mathbf{Pr} \left[(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\mathrm{cp}} \mid X = \mathbf{x}' \right], \\ \mathbf{Pr} \left[(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\mathrm{cp}} \mid Y = \mathbf{y} \right] = \mathbf{Pr} \left[(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\mathrm{cp}} \mid Y = \mathbf{y}' \right].$$

This means that $p^X_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}$ and $p^Y_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}$ are well-defined. Moreover, it holds that

(9)
$$p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^X = \mu_{\mathcal{C}}(\mathcal{F}\wedge\tau) \quad and \quad p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^Y = \mu_{\mathcal{C}\setminus\{c_0\}}(\mathcal{E}\wedge\sigma)$$

Proposition 5.2 can be proved by following the same argument as in [WY24, Proposition 4.2]. In fact, the same proof applies as long as the coupling is well-defined. Therefore, we omit the proof here.

The following proposition outlines families of linear constraints satisfied by the marginal probabilities.

Proposition 5.3. Assume Φ is satisfiable. Fix any $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{V}^{cp}$. The following properties hold for $p^X_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}$ and $p^Y_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}$:

- (1) It always holds that $p^X_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}, p^Y_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} \in [0,1]$. In particular, $p^X_{(C \setminus \{c_0\}, C, \varnothing, \varnothing, \emptyset, \varnothing)} = p^Y_{(C \setminus \{c_0\}, C, \varnothing, \emptyset, \emptyset)} = 1.$
- (2) If $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \notin \mathcal{L}^{cp}$, then the following holds: (a) If $\mathcal{F}^{\tau} \nsubseteq \mathcal{E}^{\sigma}$, let c be the smallest constraint in $\mathcal{F}^{\tau} \setminus \mathcal{E}^{\sigma}$, and define $\pi \triangleq \texttt{False}(c)$. Then,

$$\begin{split} p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{X} = & p_{(\mathcal{E}\cup\{c\},\mathcal{F},\sigma,\tau,T,\varsigma)}^{X} = \sum_{\substack{\rho \in [q]^{\mathrm{vbl}(c)} \\ (\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T \oplus c^{O},\varsigma\wedge\rho) \in \mathcal{V}^{\mathrm{cp}} \\ p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{Y} = & p_{(\mathcal{E}\cup\{c\},\mathcal{F},\sigma,\tau,T,\varsigma)}^{Y} + p_{(\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T \oplus c^{O},\varsigma\wedge\rho)}^{Y}, \\ for \ any \ \rho \in [q]^{\mathrm{vbl}(c)} \ such \ that \ (\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T \oplus c^{O},\varsigma\wedge\rho) \in \mathcal{V}^{\mathrm{cp}}. \end{split}$$

$$\begin{array}{l} (b) \ \ If \mathcal{F}^{\tau} \subseteq \mathcal{E}^{\sigma}, \ let \ c \ be \ the \ smallest \ constraint \ in \ \mathcal{E}^{\sigma} \setminus \mathcal{F}^{\tau}, \ and \ define \ \rho \ \triangleq \ \mathsf{False}(c). \ Then \ p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{X} = p_{(\mathcal{E},\mathcal{F}\cup\{c\},\sigma,\tau,T,\varsigma)}^{X} + p_{(\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\pi)}^{X}, \\ for \ any \ \pi \in [q]^{\mathrm{vbl}(c)} \ such \ that \ (\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\pi) \in \mathcal{V}^{\mathrm{cp}}; \\ p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{Y} = p_{(\mathcal{E},\mathcal{F}\cup\{c\},\sigma,\tau,T,\varsigma)}^{Y} = \sum_{\substack{\pi \in [q]^{\mathrm{vbl}(c)} \\ (\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\pi) \in \mathcal{V}^{\mathrm{cp}}}} p_{(\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\pi)}^{Y}. \end{array}$$

(3) Furthermore, it always holds that

$$p^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} \cdot \frac{|\Omega^{\mathcal{E}\wedge\sigma}|}{|\Omega^{C\setminus\{c_0\}}|} = p^{Y}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} \cdot \frac{|\Omega^{\mathcal{F}\wedge\tau}|}{|\Omega^{C}|},$$

and

$$p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^X, p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^Y \le q^{-|\mathrm{vbl}(T)\cap V_{\mathrm{good}}|} \cdot \left(1 - \mathrm{e}q^{-(1-\varepsilon_1)k}\right)^{-|\mathrm{vbl}(T)|p_1\alpha}$$

Proof. Items 1 and 2 are derived directly from Definition 5.1 and the well-definedness ensured in Proposition 5.2, both of which are straightforward to verify.

The equation in Item 3 follows from (9). From the same equation (9), we can further derive that $p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^X = \mu_C(\mathcal{F} \wedge \tau) \leq \mu_C(\tau)$ and $p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^Y = \mu_{C\setminus\{c_0\}}(\mathcal{E} \wedge \sigma) \leq \mu_{C\setminus\{c_0\}}(\sigma)$. Together with Lemma 4.11, these imply the final inequalities in Item 3.

5.2. Setting up the linear program. Next, we construct a linear program that that mimics the marginal probabilities $p^X_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}$ and $p^Y_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}$.

5.2.1. *The coupling tree.* To set up the linear program, we first construct a recursion tree for the coupling procedure Couple($C \setminus \{c_0\}, C, \emptyset, \emptyset$), truncating it when the size of the witness tree *T* exceeds *M*.

Definition 5.4 (*M*-truncated coupling tree). The *M*-truncated coupling tree, denoted $\mathcal{T} = \mathcal{T}_M(\Phi, c_0)$, is a finite rooted tree, where each node in \mathcal{T} corresponds to a tuple $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{V}^{cp}$. The tree \mathcal{T} is constructed inductively as follows:

- (1) The root of \mathcal{T} corresponds to $(C \setminus \{c_0\}, C, \emptyset, \emptyset, \emptyset)$, and has depth 0.
- (2) For i = 0, 1, ..., consider all nodes (E, F, σ, τ, T, ς) ∈ V(T) of depth i in the current tree T.
 (a) If σ violates E or τ violates F or E^σ = F^τ or |V(T)| = M, then (E, F, σ, τ, T, ς) is left as a leaf node in T.
 - (b) Otherwise, if $\mathcal{F}^{\tau} \not\subseteq \mathcal{E}^{\sigma}$, then pick the smallest $c \in \mathcal{F}^{\tau} \setminus \mathcal{E}^{\sigma}$ and add $(\mathcal{E} \cup \{c\}, \mathcal{F}, \sigma, \tau, T, \varsigma)$ as a child of $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)$ in \mathcal{T} . Furthermore, for each $\rho \in [q]^{\text{vbl}(c)}$ and $\pi = \text{False}(c)$, add $(\mathcal{E}, \mathcal{F}, \sigma \wedge \pi, \tau \wedge \rho, T \oplus c^{\mathcal{O}}, \varsigma \wedge \rho)$ as a child of $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)$ in \mathcal{T} .
 - (c) Otherwise, it holds that $\mathcal{F}^{\tau} \subseteq \mathcal{E}^{\sigma}$. Then, pick the smallest $c \in \mathcal{E}^{\sigma} \setminus \mathcal{F}^{\tau}$ and add $(\mathcal{E}, \mathcal{F} \cup \{c\}, \sigma, \tau, T, \varsigma)$ as a child of $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)$ in \mathcal{T} . Furthermore, for each $\pi \in [q]^{\text{vbl}(c)}$ and $\rho = \text{False}(c)$, add $(\mathcal{E}, \mathcal{F}, \sigma \land \pi, \tau \land \rho, T \oplus c^{O}, \varsigma \land \pi)$ as a child of $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)$ in \mathcal{T} .

Let $\mathcal L$ denote the set of leaf nodes in $\mathcal T$. We further define the following sets of leaf nodes:

- $\mathcal{L}_{\text{coup}} \triangleq \{ (\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L} \mid \mathcal{E}^{\sigma} = \mathcal{F}^{\tau} \} \text{ as the set of "coupled" leaf nodes in } \mathcal{T}; \}$
- $\mathcal{L}_{\text{trun}} \triangleq \{ (\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L} \mid |V(T)| = M \}$ as the set of "truncated" leaf nodes in \mathcal{T} ;
- $\mathcal{L}_{\text{valid}} \triangleq \mathcal{L}_{\text{trun}} \cup \mathcal{L}_{\text{coup}}$ as the set of "valid" leaf nodes in \mathcal{T} ;
- $\mathcal{L}_{invld} \triangleq \mathcal{L} \setminus \mathcal{L}_{valid}$ as the set of "invalid" leaf nodes in \mathcal{T} .

Proposition 5.5. For any satisfiable $\Phi = (V, [q], C)$, any $c_0 \in C$ and $M \ge 1$, the *M*-truncated coupling tree $\mathcal{T} = \mathcal{T}_M(\Phi, c_0)$ has a depth of at most $M\Delta k + 1$ and a branching number of at most q^{2k} , where $\Delta = \Delta(\Phi)$ is the maximum degree of G_{Φ} .

Proof. By contradiction, assume there exists a node $(\mathcal{E}', \mathcal{F}', \sigma', \tau', T', \varsigma') \in V(\mathcal{T})$ with depth $M \Delta k + 2$. We track the size of $\mathcal{E}^{\sigma} \Delta \mathcal{F}^{\tau}$ along the path from the root to $(\mathcal{E}', \mathcal{F}', \sigma', \tau', T', \varsigma')$, denoting this size by *t*. Initially, by Definition 5.6, we have t = 1. At each intermediate node $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in V(\mathcal{T})$ along the path: we either add some constraint *c* into \mathcal{E} or \mathcal{F} , reducing *t* by 1, or we assign values to σ and τ on vbl(*c*), which increases *t* by at most $k\Delta - 1$ (since at most $k\Delta$ new elements can be added into $\mathcal{E}^{\sigma} \Delta \mathcal{F}^{\tau}$, and *c* is removed from $\mathcal{E}^{\sigma} \Delta \mathcal{F}^{\tau}$). In the latter case, the size of *T* grows by one according to Proposition 4.5. Let *i* denote the number of times this latter operation is executed.

Since the depth of $(\mathcal{E}', \mathcal{F}', \sigma', \tau', T', \varsigma')$ is $M\Delta k + 2$, the above step is repeated for $M\Delta k + 2$ times. Finally, at the node $(\mathcal{E}', \mathcal{F}', \sigma', \tau', T', \varsigma')$, we still have $t = |\mathcal{E}'^{\sigma'} \Delta \mathcal{F}'^{\tau'}| \ge 0$. Therefore,

$$(k\Delta - 1) \cdot i + 1 - (M\Delta k + 2 - i) \ge 0,$$

This implies that |T'| = i > M, which contradicts the truncation condition in Item 2a of Definition 5.4. Finally, it is easy to observe that each node in \mathcal{T} has at most q^{2k} children. 5.2.2. *The linear program.* We now present the linear program, constructed on the *M*-truncated coupling tree from Definition 5.4. Each node of the tree, denoted as $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)$, is associated with two variables mimicking $p^X_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}$ and $p^Y_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}$. The linear constraints of this LP are derived from the properties listed in Proposition 5.3.

Definition 5.6 (linear program induced by the coupling). Let $\mathcal{T} = \mathcal{T}_M(\Phi, c_0)$ denote the *M*-truncated coupling tree, constructed according to Definition 5.4. Given two parameters $0 \le r_- \le r_+$, we define a linear program using variables $\hat{p}^X_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}$ and $\hat{p}^Y_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}$ for all $(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma) \in V(\mathcal{T})$:

I. Range constraints:

$$\hat{p}^{X}_{(C \setminus \{c_0\}, C, \emptyset, \emptyset, \emptyset)} = \hat{p}^{Y}_{(C \setminus \{c_0\}, C, \emptyset, \emptyset, \emptyset)} = 1; \hat{p}^{X}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}, \hat{p}^{Y}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)} \in [0, 1], \qquad \forall (\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in V(\mathcal{T}).$$

II. Non-leaf constraints: For each non-leaf node $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in V(\mathcal{T}) \setminus \mathcal{L}$: (a) If $\mathcal{F}^{\tau} \not\subseteq \mathcal{E}^{\sigma}$, let *c* be the smallest constraint in $\mathcal{F}^{\tau} \setminus \mathcal{E}^{\sigma}$ and $\pi = \mathtt{False}(c)$:

$$\begin{split} \hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} &= \hat{p}^{X}_{(\mathcal{E}\cup\{c\},\mathcal{F},\sigma,\tau,T,\varsigma)} = \sum_{\rho\in[q]^{\mathrm{vbl}(c)}} \hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\rho)}; \\ \hat{p}^{Y}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} &= \hat{p}^{Y}_{(\mathcal{E}\cup\{c\},\mathcal{F},\sigma,\tau,T,\varsigma)} + \hat{p}^{Y}_{(\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\rho)}, \quad \forall \rho \in [q]^{\mathrm{vbl}(c)}. \end{split}$$

(b) Otherwise, if $\mathcal{F}^{\tau} \subseteq \mathcal{E}^{\sigma}$, let *c* be the smallest constraint in $\mathcal{E}^{\sigma} \setminus \mathcal{F}^{\tau}$ and $\rho = \texttt{False}(c)$:

$$\hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} = \hat{p}^{X}_{(\mathcal{E},\mathcal{F}\cup\{c\},\sigma,\tau,T,\varsigma)} + \hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\pi)}, \quad \forall \pi \in [q]^{\mathrm{vbl}(c)};$$

$$\hat{p}^{Y}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} = \hat{p}^{Y}_{(\mathcal{E},\mathcal{F}\cup\{c\},\sigma,\tau,T,\varsigma)} = \sum_{\pi \in [q]^{\mathrm{vbl}(c)}} \hat{p}^{Y}_{(\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\pi)}.$$

III. *Leaf constraints*: For each leaf node $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}$: (a) If it is a coupled leaf $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{coup}$,

$$r_{-} \cdot \hat{p}^{Y}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} \leq \hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} \leq r_{+} \cdot \hat{p}^{Y}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}$$

(b) If it is an invalid leaf $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{invld}$,

$$\begin{split} \hat{p}^{Y}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} &= 0, \text{ if } \sigma \text{ violates } \mathcal{E}; \\ \hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} &= 0, \text{ if } \tau \text{ violates } \mathcal{F}. \end{split}$$

IV. *Truncation constraints*: For each truncated leaf node $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}^{\text{trun}}$:

$$p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{X}, p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{Y} \leq q^{-|\operatorname{vbl}(T) \cap V_{\operatorname{good}}|} \cdot \left(1 - \operatorname{e} q^{-(1-\varepsilon_1)k}\right)^{-|\operatorname{vbl}(T)|p_1\alpha}$$

Remark 5.7. This linear program closely resembles the one presented in [WY24], with the primary distinction being the last class of linear constraints: the truncation constraints. These constraints replace the "overflow constraints" used in the LP from [WY24]. Notably, in their design of the LP, this class of overflow constraints also distinguishes their approach from other LP-based algorithms, such as those in [Moi19, GLLZ19, JPV21, GGGY21], and is considered a key step in approaching the critical threshold in their context. In contrast, the use of truncation constraints here effectively captures the critical behavior of random instances and is essential for approaching the critical threshold in this new context.

5.3. **Analysis of the linear program.** We now establish both the correctness and efficiency of the linear program constructed earlier.

5.3.1. Performance of the LP. First, we show that the feasibility of the LP can be checked efficiently.

Proposition 5.8. Assume Condition 3.1. For any $0 \le r_- \le r_+$, the feasibility of the linear program in Definition 5.6 can be determined within $\exp(M \cdot \operatorname{poly}(k, \log q, \alpha))$ time.

This result follows directly from Proposition 5.5, as the size of the linear program is bounded by the size of the *M*-truncated coupling tree $\mathcal{T} = \mathcal{T}_M(\Phi, c_0)$, which is at most $\exp(M \cdot \operatorname{poly}(k, \log q, \alpha))$ according to the bound on maximum degree (Property 3.4) in Condition 3.1.

Next, we prove the soundness of this linear program by showing that the true values of the marginal probabilities satisfy all the linear constraints.

Lemma 5.9. Assume Φ is satisfiable. The linear program in Definition 5.6 is feasible for

$$r_{-} = r_{+} = \frac{\left|\Omega^{C \setminus \{c_{0}\}}\right|}{\left|\Omega^{C}\right|}$$

Proof. For each $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in V(\mathcal{T})$, we define the following quantities:

(10) $\hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} = \mu_{C}(\mathcal{F}\wedge\tau), \quad \hat{p}^{Y}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} = \mu_{C\setminus\{c_{0}\}}(\mathcal{E}\wedge\sigma).$

We will show that they form a feasible solution to the LP described in Definition 5.6 with the parameters $r_{-} = r_{+} = \frac{|\Omega^{C \setminus \{c_0\}}|}{|\Omega^{C}|}$. By Proposition 5.2, these quantities in (10) are consistent with the actual values of $p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{X}$, $p_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{Y}$, as defined in Definition 5.1, also extending these marginal probabilities to all nodes in $V(\mathcal{T})$ rather than just \mathcal{V}^{cp} . Note that compared to \mathcal{V}^{cp} , which contains only those $(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)$ corresponding to the final outcomes of the *M*-truncated coupling procedure, the set $V(\mathcal{T})$ also contains all $(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)$ corresponding to the intermediate steps of the coupling. We then verify that (10) satisfies all linear constraints of the LP in Definition 5.1:

we then verify that (10) satisfies an intear constraints of the Li in Definition 5.1.

- *Range and Non-leaf constraints*: These constraints hold by Items 1 and 2 of Proposition 5.3.
- *Leaf constraints:* Item III.b is a direct consequence of (10). To verify Item III.a, note that for $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{coup}$, we have $\mathcal{E}^{\sigma} = \mathcal{F}^{\tau}$, which implies that $|\Omega^{\mathcal{E} \wedge \sigma}| = |\Omega^{\mathcal{F} \wedge \tau}|$. Thus:
 - If $|\Omega^{\mathcal{E}\wedge\sigma}| = |\Omega^{\mathcal{F}\wedge\tau}| > 0$, then Item III.a follows directly from Item 3 of Proposition 5.3.
 - If $|\Omega^{\mathcal{E}\wedge\sigma}| = |\Omega^{\mathcal{F}\wedge\tau}| = 0$, then $\mu_{\mathcal{C}}(\mathcal{F}\wedge\tau) = \mu_{\mathcal{C}\setminus\{c_0\}}(\mathcal{E}\wedge\sigma) = 0$ and Item III.a holds.
- *Truncation constraints:* The constraints in Item IV. hold by Item 3 of Proposition 5.3.

At last, we show that the feasibility of the linear program implies that r_{-} and r_{+} provide respective lower and upper bounds for $\frac{|\Omega^{C \setminus \{c_0\}}|}{|\Omega^{C}|}$ with bounded multiplicative error. With this, we can apply a binary search to approximate the true value of $\frac{|\Omega^{C \setminus \{c_0\}}|}{|\Omega^{C}|}$.

Lemma 5.10. Assume Condition 3.1. If the LP in Definition 5.6 is feasible for parameters $0 \le r_{-} \le r_{+}$, then it holds that

$$\left(1-2\cdot 2^{-M}\right)r_{-} \leq \frac{|\Omega^{C\setminus\{c_0\}}|}{|\Omega^{C}|} \leq \left(1+2\cdot 2^{-M}\right)r_{+}.$$

The proof of Lemma 5.10 relies on the following claim, which provides an upper bound on the estimated average marginal probabilities for truncated nodes of the coupling tree.

Claim 5.11. Assume Condition 3.1. The following hold for the solution of the LP in Definition 5.6:

$$\frac{1}{|\Omega^{C \setminus \{c_0\}}|} \sum_{\boldsymbol{x} \in \Omega^{C \setminus \{c_0\}}} \sum_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{trun}} \\ \text{with } \mathcal{E} \land \sigma \text{ satisfied by } \boldsymbol{x}}} \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{X} \leq 2^{-M},$$
$$\frac{1}{|\Omega^{C}|} \sum_{\boldsymbol{y} \in \Omega^{C}} \sum_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{trun}} \\ \text{with } \mathcal{F} \land \tau \text{ satisfied by } \boldsymbol{y}}} \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{Y} \leq 2^{-M}.$$

Assuming this claim holds, Lemma 5.10 can be proved using the same argument as in the proof of [WY24, Lemma 4.11]. For completeness, we provide the proof here.

Proof of Lemma 5.10. Let $\hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}$ and $\hat{p}^{Y}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}$ denote a feasible solution of the linear program in Definition 5.6. First, we claim the following equations hold for this feasible solution:

(11)
$$\sum_{\substack{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in\mathcal{L}_{\text{valid}}\\\text{with }\mathcal{E}\wedge\sigma\text{ satisfied by }\mathbf{x}}} \hat{p}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{X} = 1, \quad \text{for all }\mathbf{x}\in\Omega^{C\setminus\{c_{0}\}},$$
$$\sum_{\substack{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in\mathcal{L}_{\text{valid}}\\\text{with }\mathcal{F}\wedge\tau\text{ satisfied by }\mathbf{y}}} \hat{p}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{Y} = 1, \quad \text{for all }\mathbf{y}\in\Omega^{C}.$$

These equations follow directly from verifying Definition 5.6. By summing these equations over all $x \in \Omega^{C \setminus \{c_0\}}$ and all $y \in \Omega^C$, we obtain:

2)

$$|\Omega^{C\setminus\{c_0\}}| = \sum_{\substack{\boldsymbol{x}\in\Omega^{C\setminus\{c_0\}}\\\text{with }\mathcal{E}\wedge\sigma \text{ satisfied by }\boldsymbol{x}}} \sum_{\substack{\hat{p}'(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma) \in \mathcal{L}_{\text{valid}}\\\text{with }\mathcal{F}\wedge\tau \text{ satisfied by }\boldsymbol{y}}} \hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)},$$

Thus, we can express $|\Omega^{C \setminus \{c_0\}}|$ as follows:

(1

$$(by (12)) \qquad |\Omega^{C \setminus \{c_0\}}| = \sum_{x \in \Omega^{C \setminus \{c_0\}}} \sum_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{valid}} \\ \text{with } \mathcal{E} \land \sigma \text{ satisfied by } x}} \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^X$$

$$= \sum_{x \in \Omega^{C \setminus \{c_0\}}} \sum_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{coup}} \\ \text{with } \mathcal{E} \land \sigma \text{ satisfied by } x}} \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^X$$

$$+ \sum_{x \in \Omega^{C \setminus \{c_0\}}} \sum_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{trun}} \\ \text{with } \mathcal{E} \land \sigma \text{ satisfied by } x}} \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^X$$

$$(by \text{ Claim 5.11}) \qquad \leq \sum_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{coup}}} |\Omega^{\mathcal{E} \land \sigma}| \cdot \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^X + 2^{-M} \cdot |\Omega^{C \setminus \{c_0\}}|.$$

Thus, $|\Omega^{C \setminus \{c_0\}}|$ can be bounded as:

$$\begin{aligned} \left| \Omega^{\mathcal{C} \setminus \{c_0\}} \right| &\in \left[\hat{z}^X, \ \left(1 + 2 \cdot 2^{-M} \right) \hat{z}^X \right], \\ \text{where } \hat{z}^X &\triangleq \sum_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{coup}}} \left| \Omega^{\mathcal{E} \wedge \sigma} \right| \cdot \hat{p}^X_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}. \end{aligned}$$

Similarly, $|\Omega^C|$ can also be bounded as:

$$\begin{split} \left| \Omega^{C} \right| &\in \left[\hat{z}^{Y}, \ \left(1 + 2 \cdot 2^{-M} \right) \hat{z}^{Y} \right], \\ \text{where } \hat{z}^{Y} &\triangleq \sum_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{coup}}} \left| \Omega^{\mathcal{F} \wedge \tau} \right| \cdot \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{Y}. \end{split}$$

Consequently, the ratio $|\Omega^{C \setminus \{c_0\}}| / |\Omega^C|$ is bounded as:

$$\frac{|\Omega^{C \setminus \{c_0\}}|}{|\Omega^{C}|} \leq \left(1 + 2 \cdot 2^{-M}\right) \frac{\sum\limits_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{coup}}}} |\Omega^{\mathcal{E} \wedge \sigma}| \cdot \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{X}}{\sum\limits_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{coup}}}} |\Omega^{\mathcal{F} \wedge \tau}| \cdot \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{Y}} \\ \leq \left(1 + 2 \cdot 2^{-M}\right) r_{+},$$

where the last inequality follows from the leaf constraints of the LP (Item III. in Definition 5.6).

By symmetry, we also have:

$$\frac{|\Omega^{C \setminus \{c_0\}}|}{|\Omega^{C}|} \geq \left(1 - 2 \cdot 2^{-M}\right) \frac{\sum\limits_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{coup}}}} |\Omega^{\mathcal{E} \wedge \sigma}| \cdot \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{X}}{\sum\limits_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{coup}}}} |\Omega^{\mathcal{F} \wedge \tau}| \cdot \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{Y}}{\sum\limits_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{coup}}}} |\Omega^{\mathcal{F} \wedge \tau}| \cdot \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{Y}}$$

Combining these results, we conclude:

$$\left(1-2\cdot 2^{-M}\right)r_{-} \leq \frac{|\Omega^{C\setminus\{c_0\}}|}{|\Omega^{C}|} \leq \left(1+2\cdot 2^{-M}\right)r_{+}.$$

5.3.2. Proof of the average marginal bound (Claim 5.11). To prove Claim 5.11, we introduce an auxiliary random process induced by the LP feasible solution. This random process plays a key role in the proof of Claim 5.11, and also forms the foundation for the sampling algorithm derived from the LP.

Definition 5.12 (random path induced by the LP solution). We define the following random process for generating a random path from the root to a leaf in the *M*-truncated coupling tree $\mathcal{T} = \mathcal{T}_M(\Phi, c_0)$, based on a feasible solution $\hat{p}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^X$, $\hat{p}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^Y$ to the linear program in Definition 5.6. Let $\mathfrak{X}^{\text{lp}} \sim \mu_{C \setminus \{c_0\}}$ be a random satisfying assignment. The random process starts at the root

 $(C \setminus \{c_0\}, C, \emptyset, \emptyset, \emptyset, \emptyset)$ and proceeds as follows at each non-leaf node $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)$ in \mathcal{T} :

(1) If $\mathcal{F}^{\tau} \not\subseteq \mathcal{E}^{\sigma}$, let *c* be the smallest constraint in $\mathcal{F}^{\tau} \setminus \mathcal{E}^{\sigma}$:

- If *c* is satisfied by \mathfrak{X}^{lp} , move to the child node $(\mathcal{E} \cup \{c\}, \mathcal{F}, \sigma, \tau, T, \varsigma)$.
- Otherwise, for each $\rho \in [q]^{\text{vbl}(c)}$, move to the child node $(\mathcal{E}, \mathcal{F}, \sigma \land \mathfrak{X}^{\text{lp}}_{\text{vbl}(c)}, \tau \land \rho, T \oplus$ $c^{O}, \varsigma \wedge \rho$) with probability:

$$\frac{\hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma\wedge\mathfrak{X}^{\mathrm{lp}}_{\mathrm{vbl}(c)},\tau\wedge\rho,T\oplus c^{\mathcal{O}},\varsigma\wedge\rho)}}{\hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}}$$

- (2) Otherwise, if $\mathcal{F}^{\tau} \subseteq \mathcal{E}^{\sigma}$, let *c* be the smallest constraint in $\mathcal{E}^{\sigma} \setminus \mathcal{F}^{\tau}$:
 - Move to the child node $(\mathcal{E}, \mathcal{F} \cup \{c\}, \sigma, \tau, T, \varsigma)$ with probability:

$$\frac{\hat{p}^{X}_{(\mathcal{E},\mathcal{F}\cup\{c\},\sigma,\tau,T,\varsigma)}}{\hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}};$$

• Alternatively, move to the child node $(\mathcal{E}, \mathcal{F}, \sigma \land \mathfrak{X}^{\text{lp}}_{\text{vbl}(c)}, \tau \land \rho, T \oplus c^{O}, \varsigma \land \mathfrak{X}^{\text{lp}}_{\text{vbl}(c)})$, where $\rho = \texttt{False}(c)$, with probability:

$$\frac{\hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma\wedge\mathfrak{X}^{\mathrm{lp}}_{\mathrm{vbl}(c)},\tau\wedge\rho,T\oplus c^{\mathcal{O}},\varsigma\wedge\mathfrak{X}^{\mathrm{lp}}_{\mathrm{vbl}(c)})}{\hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}}$$

Let P^{lp} denote the random path generated by this process, and let μ^{lp} denote its distribution.

By Item II. of Definition 5.6, it is straightforward to verify that the random process in Definition 5.12 generates a root-to-leaf path P^{lp} in \mathcal{T} . We also have the following probability bounds.

Lemma 5.13. Assume Φ is satisfiable. For each $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in V(\mathcal{T})$, it holds that

$$\mathbf{Pr}\left[(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\mathrm{lp}}\right]=\mu_{\mathcal{C}\setminus\{c_0\}}(\mathcal{E}\wedge\sigma)\cdot\hat{p}^X_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}.$$

Moreover, conditioned on $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\text{lp}}$, it follows that

$$\mathfrak{X}^{\text{lp}} \sim \mu_{\mathcal{E}}^{\sigma}$$

for each $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)$ such that $\Pr\left[(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\operatorname{lp}}\right] > 0.$

Proof. We prove the lemma by structural induction in the top-down manner. The induction basis is the root node $(C \setminus \{c_0\}, C, \emptyset, \emptyset, \emptyset, \emptyset)$, where the lemma follows easily from Definition 5.12.

For the induction step, we only consider the case when $\mathcal{F}^{\tau} \not\subseteq \mathcal{E}^{\sigma}$, as the complementary case with $\mathcal{F}^{\tau} \subseteq \mathcal{E}^{\sigma}$ is straightforward. Let *c* be the smallest constraint in $\mathcal{F}^{\tau} \setminus \mathcal{E}^{\sigma}$. We have the following cases:

• **Case 1**: *c* is satisfied by \mathfrak{X}^{lp} . The process transitions to $(\mathcal{E} \cup \{c\}, \mathcal{F}, \sigma, \tau, T, \varsigma)$. By the induction hypothesis, we know that $\mathfrak{X}^{\text{lp}} \sim \mu_{\mathcal{E}}^{\sigma}$. Therefore, the event that *c* is satisfied by \mathfrak{X}^{lp} occurs with probability $\mu_{\mathcal{E}}^{\sigma}(c)$. Consequently, we have

$$\Pr \left[(\mathcal{E} \cup \{c\}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\text{lp}} \right]$$

(by Definition 5.12)
$$= \Pr \left[(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\text{lp}} \right] \cdot \mu_{\mathcal{E}}^{\sigma}(c)$$

(by I.H.)
$$= \mu_{\mathcal{C} \setminus \{c_0\}} (\mathcal{E} \wedge \sigma) \cdot \hat{p}^X_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)} \cdot \mu_{\mathcal{E}}^{\sigma}(c)$$

(by the chain rule)
$$= \mu_{\mathcal{C} \setminus \{c_0\}} ((\mathcal{E} \cup \{c\}) \wedge \sigma) \cdot \hat{p}^X_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}$$

Additionally, conditioning on moving to $(\mathcal{E} \cup \{c\}, \mathcal{F}, \sigma, \tau, T, \varsigma)$, it follows that

$$\mathfrak{X}^{\mathrm{lp}} \sim \mu^{\sigma}_{\mathcal{E} \cup \{c\}},$$

thus completing the proof of this case.

• **Case 2**: *c* is violated by \mathfrak{X}^{lp} . Let $\pi = \mathfrak{X}^{\text{lp}}_{\text{vbl}(c)}$, then we have $\pi = \text{False}(c)$. By the induction hypothesis, we know that $\mathfrak{X}^{\text{lp}} \sim \mu_{\mathcal{E}}^{\sigma}$. Therefore, the event that *c* is violated by \mathfrak{X}^{lp} occurs with probability

$$\mu_{\mathcal{E}}^{\sigma}(\neg c) = \mu_{\mathcal{E}}^{\sigma}(\pi).$$

In this case, for each $\rho \in [q]^{\text{vbl}(c)}$, the process transitions to $(\mathcal{E}, \mathcal{F}, \sigma \land \pi, \tau \land \rho, T \oplus c^{\mathcal{O}}, \varsigma \land \rho)$ with probability $\frac{\hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma\land\pi,\tau\land\rho,T\oplus c^{\mathcal{O}},\varsigma\land\rho)}}{\hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}}$. Therefore, for each $\rho \in [q]^{\text{vbl}(c)}$, we have

$$\begin{aligned} & \mathbf{Pr}\left[(\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\rho)\in P^{\mathrm{lp}}\right] \\ (\text{by Definition 5.12}) &= \mathbf{Pr}\left[(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\mathrm{lp}}\right]\cdot\mu_{\mathcal{E}}^{\sigma}(\pi)\cdot\frac{\hat{p}_{(\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\rho)}^{X}}{\hat{p}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{X}} \\ & (\text{by I.H.}) &= \mu_{C\setminus\{c_0\}}(\mathcal{E}\wedge\sigma)\cdot\hat{p}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{X}\cdot\mu_{\mathcal{E}}^{\sigma}(\pi)\cdot\frac{\hat{p}_{(\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\rho)}^{X}}{\hat{p}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{X}} \\ & (\text{by the chain rule}) &= \mu_{C\setminus\{c_0\}}(\mathcal{E}\wedge\sigma\wedge\pi)\cdot\hat{p}_{(\mathcal{E},\mathcal{F},\sigma\wedge\pi,\tau\wedge\rho,T\oplus c^{O},\varsigma\wedge\rho)}^{X}. \end{aligned}$$

This concludes the proof for the case when $\mathcal{F}^{\tau} \not\subseteq \mathcal{E}^{\sigma}$. As we discussed earlier, it proves the lemma. \Box

The following corollary follows from Lemma 5.13 by the definition of conditional probability. **Corollary 5.14.** Assume Φ is satisfiable. For each $\mathbf{x} \in \Omega^{C \setminus \{c_0\}}$ and each $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in V(\mathcal{T})$,

$$\Pr\left[(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\operatorname{lp}}\mid\mathfrak{X}^{\operatorname{lp}}=\boldsymbol{x}\right] = \begin{cases} 0 & \text{if }\boldsymbol{x}\notin\Omega^{\mathcal{E}\wedge\sigma},\\ \hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} & \text{if }\boldsymbol{x}\in\Omega^{\mathcal{E}\wedge\sigma}. \end{cases}$$

Proof. For each $x \in \Omega$ and each $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in V(\mathcal{T})$, we have

(13)
$$\mathbf{Pr}\left[(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\mathrm{lp}}\mid \mathfrak{X}^{\mathrm{lp}}=\mathbf{x}\right] = \frac{\mathbf{Pr}\left[(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\mathrm{lp}}\wedge\mathfrak{X}^{\mathrm{lp}}=\mathbf{x}\right]}{\mathbf{Pr}\left[\mathfrak{X}^{\mathrm{lp}}=\mathbf{x}\right]}$$

(by Definition 5.12)
$$=|\Omega^{C\setminus\{c_0\}}|\cdot\mathbf{Pr}\left[(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\mathrm{lp}}\wedge\mathfrak{X}^{\mathrm{lp}}=\mathbf{x}\right]$$

If $x \notin \Omega^{\mathcal{E} \wedge \sigma}$, then by Lemma 5.13, the above expression equals 0. Therefore, we assume $x \in \Omega^{\mathcal{E} \wedge \sigma}$. By Lemma 5.13, conditioning on the event $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)$, the distribution of σ is uniform over all assignments in $\Omega^{\mathcal{E}\wedge\sigma}$. Thus, we have

$$\begin{aligned} \mathbf{Pr}\left[(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\mathrm{lp}} \wedge \mathfrak{X}^{\mathrm{lp}} = \mathfrak{x} \right] = \mathbf{Pr}\left[(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\mathrm{lp}} \right] \cdot \frac{1}{|\Omega^{\mathcal{E} \wedge \sigma}|} \\ (\mathrm{by \ Lemma \ 5.13}) &= \frac{|\Omega^{\mathcal{E} \wedge \sigma}|}{|\Omega^{C \setminus \{c_0\}}|} \cdot \hat{p}^X_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)} \cdot \frac{1}{|\Omega^{\mathcal{E} \wedge \sigma}|} \\ &= \frac{1}{|\Omega^{C \setminus \{c_0\}}|} \cdot \hat{p}^X_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}, \end{aligned}$$

combining this with (13) completes the proof of the corollary.

We are now ready to prove Claim 5.11.

Proof of Claim 5.11. We will prove the first inequality; the second inequality can be proved by following a similar approach.

First, we verify the following identity:

(14)

$$\frac{1}{|\Omega^{C \setminus \{c_0\}}|} \sum_{\substack{\boldsymbol{x} \in \Omega^{C \setminus \{c_0\}} \\ \text{with } \mathcal{E} \land \sigma}} \sum_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{trun}} \\ \text{with } \mathcal{E} \land \sigma}} \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{\mathcal{X}} \\
= \sum_{\substack{\boldsymbol{x} \in \Omega^{C \setminus \{c_0\}} \\ \text{with } \mathcal{E} \land \sigma}} \sum_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{trun}} \\ \text{with } \mathcal{E} \land \sigma}} \frac{\Pr\left[(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\text{lp}}\right]}{|\Omega^{\mathcal{E} \land \sigma}|} \\
= \sum_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{trun}}} \Pr\left[(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\text{lp}}\right],$$

where the first equality follows from Lemma 5.13 and the fact that for each $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in V(\mathcal{T})$, any assignment \mathbf{x} that satisfies $\mathcal{E} \wedge \sigma$ must also satisfy $\mathcal{C} \setminus \{c_0\}$. The second equality follows from this same fact and the exchange of the order of summation.

Combining these results, we have the following:

$$\begin{split} &\sum_{\substack{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in\mathcal{L}_{trun}}} \Pr\left[(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\mathrm{lp}}\right] \\ &\leq \sum_{T'\in\mathbb{T}_{M}^{c_{0}}} \sum_{\substack{\varsigma'\in[q]^{\mathrm{vbl}(T')}}} \sum_{\substack{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\mathrm{lp}}\\T=T'\wedge\varsigma=\varsigma'}} \Pr\left[(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\mathrm{lp}}\right] \\ &\leq \sum_{T'\in\mathbb{T}_{M}^{c_{0}}} \sum_{\substack{\varsigma'\in[q]^{\mathrm{vbl}(T')}}} \sum_{\substack{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\mathrm{lp}}\\T=T'\wedge\varsigma=\varsigma'}} \sum_{\substack{x\in\Omega^{\mathcal{E}\wedge\sigma}}} \Pr\left[\mathfrak{X}^{\mathrm{lp}}=\mathbf{x}\right] \cdot \Pr\left[(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\mathrm{lp}} \mid \mathfrak{X}^{\mathrm{lp}}=\mathbf{x}\right] \\ &= \sum_{T'\in\mathbb{T}_{M}^{c_{0}}} \sum_{\substack{\varsigma'\in[q]^{\mathrm{vbl}(T')}}} \sum_{\substack{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in P^{\mathrm{lp}}\\T=T'\wedge\varsigma=\varsigma'}} \sum_{\substack{x\in\Omega^{\mathcal{E}\wedge\sigma}}} \Pr\left[\mathfrak{X}^{\mathrm{lp}}=\mathbf{x}\right] \cdot \hat{p}^{X}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}. \end{split}$$

Here, the last equality follows directly from Corollary 5.14.

Note that by Lemma 4.7, fixing the witness tree *T* and the witness assignment ς uniquely identifies a node $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\text{lp}}$ satisfying $T = T' \land \varsigma = \varsigma'$. We then denote this unique node as $N(T', \sigma')$, and the partial assignment σ in this unique node as $\sigma(T', \varsigma')$. Therefore, we further have

$$\begin{split} \sum_{T' \in \mathbb{T}_{M}^{C_{0}}} \sum_{\varsigma' \in [q]^{\mathrm{vbl}(T')}} \sum_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in P^{\mathrm{lp}}} \sum_{\mathbf{x} \in \Omega^{\mathcal{E} \wedge \sigma}} \mathbf{Pr} \left[\mathbf{\mathfrak{X}}^{\mathrm{lp}} = \mathbf{x} \right] \cdot \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{X} \end{split}$$

$$(\star) &\leq \sum_{T' \in \mathbb{T}_{M}^{C_{0}}} \sum_{\varsigma' \in [q]^{\mathrm{vbl}(T')}} \mathbf{Pr} \left[\mathbf{\mathfrak{X}}^{\mathrm{lp}}_{\mathrm{vbl}(T')} = \sigma(T', \varsigma') \right] \cdot \hat{p}_{N(T', \sigma')}^{X}$$

$$(\star) &\leq \sum_{T' \in \mathbb{T}_{M}^{C_{0}}} \sum_{\varsigma' \in [q]^{\mathrm{vbl}(T')}} \mathbf{Pr} \left[\mathbf{\mathfrak{X}}^{\mathrm{lp}}_{\mathrm{vbl}(T')} = \sigma(T', \varsigma') \right] \cdot q^{-|\mathrm{vbl}(T') \cap V_{\mathrm{good}}|} \cdot \left(1 - \mathrm{e}q^{-(1-\varepsilon_{1})k} \right)^{-|\mathrm{vbl}(T')| p_{1}\alpha}$$

$$(\bullet) &\leq \sum_{T' \in \mathbb{T}_{M}^{C_{0}}} \sum_{\varsigma' \in [q]^{\mathrm{vbl}(T')}} q^{-2|\mathrm{vbl}(T') \cap V_{\mathrm{good}}|} \cdot \left(1 - \mathrm{e}q^{-(1-\varepsilon_{1})k} \right)^{-2|\mathrm{vbl}(T')| p_{1}\alpha}$$

$$(15) &\leq \sum_{T' \in \mathbb{T}_{M}^{C_{0}}} q^{|\mathrm{vbl}(T')|} \cdot q^{-2|\mathrm{vbl}(T') \cap V_{\mathrm{good}}|} \cdot \left(1 - \mathrm{e}q^{-(1-\varepsilon_{1})k} \right)^{-2|\mathrm{vbl}(T')| p_{1}\alpha}$$

$$(\Delta) &\leq 2^{-M}.$$

The (\star) inequality is derived using the argument above. The (\blacktriangle) inequality follows from Item IV. of Definition 5.6. The (\blacksquare) inequality follows from Lemma 4.11. Finally, the (\triangle) inequality holds for the chosen parameters in Condition 3.1 and was already established in the proof of Theorem 4.1; in particular, (15) is exactly (7) and hence the same proof applies.

5.4. **Sampling and counting via linear programming.** In this subsection, we apply the linear program in Definition 5.6 to derive sampling and counting algorithms and prove Theorem 3.2. The proofs essentially follow those in [WY24, Section 5], and we include for completeness.

We begin with the counting algorithm, which directly follows from the analysis of the linear program presented in the previous subsection.

Proof of the counting part of Theorem 3.2. Let $C = \{c_1, c_2, ..., c_m\}$. Note that $km = n\alpha$. For each $0 \le i \le m$, define $C_i = \{c_1, c_2, ..., c_i\}$, $\Phi_i = (V, [q], C_i)$. In particular, $\Phi_m = \Phi$. By applying a constraint-wise self-reduction, we decompose $Z(\Phi_m)$ into the following telescopic product:

(16)
$$Z(\Phi_m) = Z(\Phi_0) \cdot \frac{Z(\Phi_m)}{Z(\Phi_0)} = Z(\Phi_0) \cdot \prod_{i=1}^m \frac{Z(\Phi_i)}{Z(\Phi_{i-1})}$$

Note that Φ_0 is a trivial CSP formula, so $Z_{\Phi_0} = q^{|V|}$. Moreover, and crucially, observe that each Φ_i satisfies Condition 3.1 for $0 \le i \le m$, assuming the original CSP formula Φ satisfies Condition 3.1. Setting:

$$M=1+\log\frac{4m}{\varepsilon},$$

we use the LP in Definition 5.6 with binary search to approximate each ratio $\frac{Z(\Phi_i)}{Z(\Phi_{i-1})} = \frac{|\Omega^{C_i}|}{|\Omega^{C_{i-1}}|}$ within a multiplicative error of $\frac{\varepsilon}{4m}$. According to Proposition 5.8 and lemmas 5.9 and 5.10, this can be done in time $O\left(\left(\frac{n}{\varepsilon}\right)^{\text{poly}(k,\log q,\alpha)}\right)$. Therefore, we obtain an estimate of $Z(\Phi) = Z(\Phi_m)$ within a total multiplicative error of ε in time $O\left(\left(\frac{n}{\varepsilon}\right)^{\text{poly}(k,\log q,\alpha)}\right)$, as required.

For the sampling part, we establish a constraint-wise self-reduction for sampling by constructing a *dynamic sampler*. Given an atomic CSP formula $\Phi = (V, [q], C)$ and a constraint $c \in C \setminus \{c_0\}$, it dynamically updates a current sample $\sigma^{\text{in}} \sim \mu_{C \setminus \{c_0\}}$ to a new sample $\sigma^{\text{out}} \sim \mu_C$. This dynamic sampler is derived from the random process in Definition 5.12, which was initially introduced to analyze the linear program.

Definition 5.15 (dynamic sampler). The algorithm takes as input a CSP $\Phi = (V, [q], C)$, a constraint $c_0 \in C$, and an error bound $\varepsilon \in (0, 1)$. Additionally, the algorithm is also given access to an assignment $\sigma^{\text{in}} \in [q]^V$ that satisfies $\Phi^{c_0} = (V, [q], C \setminus \{c_0\})$.

The algorithm proceeds as follows to update σ^{in} to a new assignment $\sigma^{\text{out}} \in [q]^V$:

• Set the truncation threshold:

$$M = 1 + \log \frac{4}{\varepsilon}.$$

Construct the LP on the *M*-truncated coupling tree $\mathcal{T} = \mathcal{T}_M(\Phi, c_0)$, as described in Definition 5.6.

- Use binary search to find an interval $[r_-, r_+]$ such that $r_- \ge \frac{4+\varepsilon}{4+2\varepsilon}r_+$ and the LP remains feasible for parameters r_- and r_+ . Let $\left\{\hat{p}^X_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}, \hat{p}^Y_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}\right\}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)\in V(\mathcal{T})}$ be the corresponding LP feasible solution.
- Simulate the random process from Definition 5.12 using the feasible solution obtained above, replacing the random assignment \mathfrak{X}^{lp} (used as the random seed for the process) with σ^{in} . Let $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}$ be the random leaf reached by this process.
- If the leaf node (ε, F, σ, τ, T, ς) ∉ L_{coup}, then set σ^{out} ∈ [q]^V to be an arbitrary satisfying assignment of Φ. Otherwise, update σⁱⁿ by modifying the assigned variables according to τ, i.e., set σ^{out} ← τ ∧ σⁱⁿ_{V\Λ(τ)}.

We will prove the following lemma, which shows that the new sample produced by the dynamic sampler approximates the target distribution μ_C .

Lemma 5.16. Assume Condition 3.1 and that $\sigma^{\text{in}} \sim \mu_{C \setminus \{c_0\}}$. Then, the assignment σ^{out} produced by the dynamic sampler in Definition 5.15 follows a distribution μ^{out} that satisfies

$$d_{\mathrm{TV}}(\mu^{\mathrm{out}},\mu_{\mathcal{C}}) \leq \varepsilon.$$

With Lemma 5.16, we can complete the proof of Theorem 3.2.

Proof of the sampling part of Theorem 3.2. Let $C = \{c_1, c_2, ..., c_m\}$. Note that $km = n\alpha$. For each $0 \le i \le m$, define $C_i = \{c_1, c_2, ..., c_i\}$, $\Phi_i = (V, [q], C_i)$ and let μ_{Φ_i} denote the uniform distribution over satisfying assignments of Φ_i . Let $\sigma_0, \sigma_1, ..., \sigma_m \in [q]^V$ be constricted as follows:

- Sample $\sigma_0 \sim \mathcal{P}$, the uniform product distribution.
- For each $1 \le i \le m$, use the dynamic sampler from Definition 5.15 to generate σ_i from σ_{i-1} , with the input formula $\Phi_i = (V, [q], C_i)$, input constraint c_i , error bound $\frac{\varepsilon}{m}$, and current sample σ_{i-1} . If σ_i is not a solution to Φ_i at any step $0 \le i \le m$, output an arbitrary final assignment instead.

By applying induction along with Lemma 5.16, we can verify that for each $0 \le i \le m$, the distribution μ_i of σ_i satisfies

$$d_{\mathrm{TV}}(\mu_i, \mu_{C_i}) \leq \frac{i}{m} \cdot \varepsilon.$$

The total running time is bounded by $O\left(\left(\frac{n}{\varepsilon}\right)^{\text{poly}(k,\log q,\alpha)}\right)$ according to Proposition 5.8, which accounts for the overhead of building and solving the linear program. Thus, the theorem is proved.

We then finish this section by proving Lemma 5.16.

Proof of Lemma 5.16. According to Definition 5.15 and Lemma 5.13, for each $y \in \Omega^C$, we have

$$\mathbf{Pr}\left[\sigma^{\text{out}} = \mathbf{y}\right] = \sum_{\substack{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma) \in \mathcal{L}_{\text{coup}} \\ \text{with } \mathcal{F} \land \tau \text{ satisfied by } \mathbf{y}}} \mu_{\mathcal{C} \backslash \{c_0\}}(\mathcal{E} \land \sigma) \cdot \hat{p}^X_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)} \cdot \frac{1}{|\Omega^{\mathcal{E} \land \sigma}|}$$

(by Item III.a of Definition 5.6)

$$(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma) \in \mathcal{L}_{coup}$$
with $\mathcal{F} \wedge \tau$ satisfied by \mathbf{y}
(6) $\geq r_{-} \cdot \sum_{\substack{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma) \in \mathcal{L}_{coup}\\ \text{with }\mathcal{F} \wedge \tau \text{ satisfied by }\mathbf{y}} \mu_{C \setminus \{c_0\}} (\mathcal{E} \wedge \sigma) \cdot \hat{p}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{Y} \cdot \frac{1}{|\Omega^{\mathcal{E} \wedge \sigma}|}$

$$= r_{-} \cdot \frac{1}{|\Omega^{C \setminus \{c_0\}}|} \cdot \sum_{\substack{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma) \in \mathcal{L}_{coup}\\ \text{with }\mathcal{F} \wedge \tau \text{ satisfied by }\mathbf{y}} \hat{p}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{Y}.$$

Here, the first equality holds because $\mathcal{E}^{\sigma} = \mathcal{F}^{\tau}$ for each $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{coup}}$, meaning each solution in $\Omega^{\mathcal{F}\wedge\tau}$ is generated with equal probability $\frac{1}{|\Omega^{\mathcal{F}\wedge\tau}|} = \frac{1}{|\Omega^{\mathcal{E}\wedge\sigma}|}$ as established in Lemma 5.13. The last equality follows from the identity $\mu_{C\setminus\{c_0\}}(\mathcal{E}\wedge\sigma) = \frac{|\Omega^{\mathcal{E}\wedge\sigma}|}{|\Omega^{C\setminus\{c_0\}}|}$, because $\mathcal{E}\wedge\sigma \implies C\setminus\{c_0\}$ for each $(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in V(\mathcal{T})$, which holds by induction.

Therefore, we can construct a distribution v over $[q]^V$ such that the measure of each $y \in \Omega^C$ is

(17)
$$\nu(\mathbf{y}) = \frac{1}{|\Omega^{C}|} \cdot \sum_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\text{coup}} \\ \text{with } \mathcal{F} \land \tau \text{ satisfied by } \mathbf{y}}} \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{Y}$$

and meanwhile the total variation distance between v and the distribution μ^{out} of σ^{out} is bounded as:

(18)
$$d_{\mathrm{TV}}\left(\boldsymbol{\nu},\boldsymbol{\mu}^{\mathrm{out}}\right) \leq \left|\frac{1}{|\boldsymbol{\Omega}^{C}|} - \boldsymbol{r}_{-} \cdot \frac{1}{|\boldsymbol{\Omega}^{C \setminus \{c_{0}\}}|}\right| \sum_{\boldsymbol{y} \in \boldsymbol{\Omega}^{C}} \sum_{\substack{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma) \in \mathcal{L}_{\mathrm{coup}} \\ \mathrm{with} \ \mathcal{F} \wedge \tau \text{ satisfied by } \boldsymbol{y}}} \hat{p}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{Y}$$

Given that $\left(1 - \frac{\varepsilon}{4}\right)r_{-} \cdot \frac{|\Omega^{C \setminus \{c_0\}}|}{|\Omega^{C}|} \le \left(1 + \frac{\varepsilon}{4}\right)r_{+}$ and $r_{-} \ge \frac{4+\varepsilon}{4+2\varepsilon}r_{+}$ are ensured by the process, we obtain

(19)
$$\left(1 - \frac{\varepsilon}{2}\right) \cdot \frac{|\Omega^{C \setminus \{c_0\}}|}{|\Omega^{C}|} \le r_{-} \le \left(1 + \frac{\varepsilon}{2}\right) \cdot \frac{|\Omega^{C \setminus \{c_0\}}|}{|\Omega^{C}|}$$

Combining (18) and (19), we have

(20)
$$d_{\mathrm{TV}}(\nu,\mu^{\mathrm{out}}) \leq \frac{\varepsilon}{2} \cdot \frac{1}{|\Omega^{C}|} \sum_{\boldsymbol{y} \in \Omega^{C}} \sum_{\substack{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma) \in \mathcal{L}_{\mathrm{coup}} \\ \mathrm{with } \mathcal{F} \wedge \tau \text{ satisfied by } \boldsymbol{y}}} \hat{p}_{(\mathcal{E},\mathcal{F},\sigma,\tau,T,\varsigma)}^{Y} \leq \frac{\varepsilon}{2}.$$

Now, consider the total variation distance between ν and μ_C , the uniform distribution over Ω^C . By (17) and (11), for each $\mathbf{y} \in \Omega^C$ we have $\nu(\mathbf{y}) \leq \frac{1}{|\Omega^C|} = \mu_C(\mathbf{y})$. Thus, we have

$$d_{\mathrm{TV}}(\mu_{\mathcal{C}}, \nu) = \sum_{\mathbf{y} \in \Omega^{\mathcal{C}}} \left(\mu_{\mathcal{C}}(\mathbf{y} - \nu(\mathbf{y})) \right)$$
$$= \sum_{\mathbf{y} \in \Omega^{\mathcal{C}}} \left(\frac{1}{|\Omega^{\mathcal{C}}|} - \frac{1}{|\Omega^{\mathcal{C}}|} \cdot \sum_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\mathrm{coup}} \\ \mathrm{with} \ \mathcal{F} \land \tau \ \mathrm{satisfied} \ \mathrm{by} \ \mathbf{y}}} \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{Y} \right)$$
$$(\mathrm{by} (11)) = \frac{1}{|\Omega^{\mathcal{C}}|} \sum_{\substack{\mathbf{y} \in \Omega^{\mathcal{C}} \\ \mathrm{with} \ \mathcal{F} \land \tau \ \mathrm{satisfied} \ \mathrm{by} \ \mathbf{y}}} \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{Y} \sum_{\substack{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma) \in \mathcal{L}_{\mathrm{coup}} \\ \mathrm{with} \ \mathcal{F} \land \tau \ \mathrm{satisfied} \ \mathrm{by} \ \mathbf{y}}} \hat{p}_{(\mathcal{E}, \mathcal{F}, \sigma, \tau, T, \varsigma)}^{Y}$$
$$(\mathrm{by} \ \mathrm{Claim} \ 5.11) \leq \frac{\mathcal{E}}{\alpha}.$$

Combining with (20) and by triangle inequality, we conclude

$$d_{\mathrm{TV}}(\mu_C, \mu^{\mathrm{out}}) \le d_{\mathrm{TV}}(\mu_C, \nu) + d_{\mathrm{TV}}(\nu, \mu^{\mathrm{out}}) = \varepsilon.$$

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Appendix A. Proofs of structural properties

In this section, we prove Lemma 3.13.

The first condition of nice hypergraphs, i.e., H_{Φ} is in $\mathcal{H}_{\leq k}$ and has density α , is trivial. Then Property 3.4 follows from a classical result [RS98, Theorem 1].

Lemma A.1. With probability 1 - o(1/n) over the random formula $\Phi = \Phi(k, n, m)$ with density α , H_{Φ} satisfies Property 3.4, namely, the maximum degree of variables is at most $4k\alpha + 6\log n$.

Similar to this lemma, the following proposition, which bounds the number of high-degree vertices in H_{Φ} , also follows from the classical result of the balls-and-bins model.

Proposition A.2. Assume $k \ge 2$, $q \ge 2$ and $\alpha \le q^k$ are constants. Let p_1 be a parameter satisfying $p_1 \ge 4k$. Then with probability 1 - o(1/n) over the random formula Φ , for $H_{\Phi} = (V, \mathcal{E})$, we have

$$|\{v \in V \mid \deg(v) > p_1 \alpha\}| \le e^{-k} \alpha^{-2} n$$
,

i.e., $|\text{HD}(V)| \le e^{-k} \alpha^{-2} n$.

Proof. The degrees of the variables in Φ distribute as a balls-and-bins experiment with km balls and n bins. Let $D_1, \ldots, D_n \sim \text{Pois}(k\alpha)$ be n independent Poisson random variables with parameter $k\alpha$. Then the degrees of the variables in Φ has the same distribution as $\{D_1, \ldots, D_n\}$ conditioned on the event $\mathcal{E}_{n,m}$ that $\sum_{i=1}^n D_i = km$ [MU17, Chapter 5.4]. Note that $\sum_{i=1}^n D_i$ is a Poisson random variable with parameter $k\alpha n = km$. Thus

$$\mathbf{Pr}\left[\mathcal{E}_{n,m}\right] = e^{-km} \cdot \frac{(km)^{km}}{(km)!} \ge \frac{1}{\sqrt{2\pi km}} = \frac{1}{\sqrt{2\pi km}}$$

For any fixed $i \in [n]$, we have

$$\mathbf{Pr}\left[D_i \ge p_1\alpha\right] = \mathbf{Pr}\left[\operatorname{Pois}(k\alpha) \ge p_1\alpha\right] \le \frac{\mathrm{e}^{-k\alpha}(\mathrm{e}k\alpha)^{p_1\alpha}}{(p_1\alpha)^{p_1\alpha}} \le \mathrm{e}^{-k\alpha}(\mathrm{e}/4)^{4k\alpha} \le \mathrm{e}^{-k\alpha}2^{-2k\alpha} \le 2^{-3k\alpha}.$$

Define $U = \{i \in [n] \mid D_i \ge p_1 \alpha\}$. Then by Chernoff-Hoeffding bound, we obtain that

$$\Pr\left[|U| > e^{-k} \alpha^{-2} n\right] \le \Pr\left[|U| - \mathbb{E}[|U|] > e^{-k+1} \alpha^{-2} n\right] < e^{-2e^{-2k+2} \alpha^{-4} n} = o(1/n^2),$$

which further gives that

$$\Pr\left[|\{v \in V(H_{\Phi}) \mid \deg(v) > p_{1}\alpha\}| \ge e^{-k}\alpha^{-2}n\right]$$

=
$$\Pr\left[|U| \ge e^{-k}\alpha^{-2}n \mid \mathcal{E}_{n,m}\right]$$

$$\le \sqrt{2\pi k\alpha n} \cdot o(1/n^{2}) = o(1/n) .$$

Now we show that H_{Φ} satisfies Property 3.5 and Property 3.6 with probability 1 - o(1/n). For Property 3.5, we present Lemma A.3 and its corollary, which are adapted from [HWY23, Proposition 3.3 & 3.4]. For Property 3.6, we have Lemma A.6, which is adapted from [GGGY21, Lemma 8.6] and [HWY23, Proposition 3.5], but gives a better bound.

Lemma A.3. For any fixed k and α , if $\eta k \ge 4$, $\rho < 1$ and $e(\rho k \alpha)^{\eta} \le 1$, then with probability 1 - o(1/n) over the choice of random k-SAT formula $\Phi = \Phi(k, n, m)$ with density α , H_{Φ} satisfies Property 3.5.

Proof. Let $\ell \leq \rho m$ and fix $r = \lfloor (1 - \eta)k\ell \rfloor$. For any $U \subseteq V$ of size r and any ℓ edges e_1, \ldots, e_ℓ , the probability that $e_i \subseteq U$ for every i is

$$\left(\frac{r}{n}\right)^{k\ell} \leq \left(\frac{(1-\eta)k\ell}{n}\right)^{k\ell}.$$

Thus, let \mathcal{E}_{ℓ} be the event that there exists $U \subseteq V$ of size r and ℓ edges e_1, \ldots, e_{ℓ} such that $e_i \subseteq U$ for every i. We obtain that

$$\begin{aligned} \Pr\left[\mathcal{E}_{\ell}\right] &\leq \binom{n}{r}\binom{m}{\ell}\binom{(1-\eta)k\ell}{n}^{k\ell} \leq \left(\frac{\mathrm{e}n}{r}\right)^{r}\left(\frac{\mathrm{e}m}{\ell}\right)^{\ell}\left(\frac{(1-\eta)k\ell}{n}\right)^{k\ell} \\ &\leq \left(\frac{\mathrm{e}n}{(1-\eta)k\ell}\right)^{(1-\eta)k\ell}\left(\frac{\mathrm{e}m}{\ell}\right)^{\ell}\left(\frac{(1-\eta)k\ell}{n}\right)^{k\ell} \\ &\leq \left(\frac{\mathrm{e}^{(1-\eta)k+1}((1-\eta)k\ell)^{\eta k}m}{n^{\eta k}\ell}\right)^{\ell} \\ &\leq \left(\frac{\mathrm{e}^{k}k^{\eta k}\alpha n\ell^{\eta k}}{n^{\eta k}\ell}\right)^{\ell} \leq \left((\mathrm{e}k^{\eta})^{k}\alpha\left(\frac{\ell}{n}\right)^{\eta k-1}\right)^{\ell}.\end{aligned}$$

If $\ell \leq n^{1/3}$, we have

$$\mathbf{Pr}\left[\mathcal{E}_{\ell}\right] \leq (\mathbf{e}k^{\eta})^k \alpha n^{-2(\eta k-1)/3} \leq (\mathbf{e}k^{\eta})^k \alpha n^{-2}$$

as long as $\eta k \ge 4$. If $n^{1/3} \le \ell \le \rho m$, noting that $(\rho \alpha)^{\eta k - 1} \le 1/(e^k k^{\eta k} \rho \alpha)$, we have

$$\mathbf{Pr}\left[\mathcal{E}_{\ell}\right] \leq \left(\left(\mathrm{e}k^{\eta}\right)^{k} \alpha \left(\rho \alpha\right)^{\eta k-1}\right)^{n^{1/3}} \leq \rho^{n^{1/3}} \leq n^{-3}$$

Therefore, by the union bound, the probability that there exists $\ell \leq \rho m$ edges e_1, \ldots, e_ℓ where

$$\left|\bigcup_{i=1}^{\ell} e_i\right| \le (1-\eta)k\ell$$

is at most

$$\sum_{\ell=1}^{\rho m} \Pr\left[\mathcal{E}_{\ell}\right] \le \frac{(ek^{\eta})^k \alpha n^{1/3}}{n^2} + n^{-2} = o(1/n) \,.$$

Corollary A.4. Let $H = (V, \mathcal{E}) \in \mathcal{H}_{\leq k}$ be a hypergraph satisfying (η, ρ) -edge expansion. Then for any $\eta < b \leq 1$ and $V' \subseteq V$ of size less than $(b - \eta)k\rho |\mathcal{E}|$, it holds that

 $|V'| \ge (b-\eta)k \left| \{e \in \mathcal{E} \mid |e \cap V'| \ge bk\} \right| \,,$

namely, the number of hyperedges e such that $|e \cap V'| \ge bk$ is at most $\lfloor |V'| / ((b - \eta)k) \rfloor$.

Proof. For the sake of contradiction, assume that there is a set $V' \subseteq V$ of size $|V'| < (b - \eta)k\rho m$ and $\ell = \lfloor |V'|/((b - \eta)k) \rfloor + 1$ hyperedges e_1, \ldots, e_ℓ such that each hyperedge contains at least bk vertices in V'. Then it is clear that $|V'| < (b - \eta)k\ell$ and thus

$$\left|\bigcup_{i=1}^{\ell} e_i\right| \le (1-b)k\ell + |V'| < (1-\eta)k\ell.$$

Since $\ell \leq \rho m$, it contradicts with Lemma A.3.

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Proposition A.5 ([GGGY21, Lemma 8.5]). Let U be any subset of indices of clauses in Φ and T be any tree on the vertex set U. Then the probability that T is a subgraph of G_{Φ} is at most $(k^2/n)^{|U|-1}$.

Lemma A.6. Suppose $\alpha \leq 2^k$. With probability 1 - o(1/n) over the choice of random k-SAT formula $\Phi = \Phi(k, n, m)$ with fixed density α , H_{Φ} satisfies Property 3.6, namely, for every clause c in Φ and $\ell \geq 1$, there are at most $n^3(ek^2\alpha)^{\ell}$ many connected sets of clauses in G_{Φ} that contains c and has size ℓ .

Proof. If $\ell = 1$, this lemma is trivial. Now we assume $\ell \ge 2$. Let *c* be an arbitrary clause in Φ and *U* be a set of clauses of size ℓ where $c \in U$. Let T_U be the set of all trees with vertex set *U*. By standard results, we have $|T_U| = \ell^{\ell-2}$. In addition, any fixed tree $T \in T_U$ is a subgraph of G_{Φ} with probability at most $(k^2/n)^{\ell-1}$, by Proposition A.5. Thus, the union bound gives that

$$\Pr[G_{\Phi}[U] \text{ is connected}] \leq \ell^{\ell-2} (k^2/n)^{\ell-1}$$

Let $Z_{\ell,c}$ be the number of connected sets of clauses with size ℓ containing c. Then, we have

$$\begin{split} \mathbf{E}[Z_{\ell,c}] &= \sum_{U:c \in U, |U|=\ell} \mathbf{Pr} \left[G_{\Phi}[U] \text{ is connected} \right] \\ &\leq \binom{m-1}{\ell-1} \cdot \ell^{\ell-2} \cdot \left(\frac{k^2}{n}\right)^{\ell-1} \\ &\leq \frac{(\mathrm{e}m)^{\ell-1}}{(\ell-1)^{\ell-1}} \ell^{\ell-2} \left(\frac{k^2}{n}\right)^{\ell-1} \\ &= \frac{(\mathrm{e}k^2\alpha)^{\ell-1}}{\ell} \left(\frac{\ell}{\ell-1}\right)^{\ell-1} \\ &\leq \frac{\mathrm{e}}{\ell} \cdot (\mathrm{e}k^2\alpha)^{\ell-1} \,. \end{split}$$

By Markov's inequality, it further implies that

$$\mathbf{Pr}\left[Z_{\ell,c} \ge n^3 (\mathbf{e}k^2 \alpha)^\ell\right] \le \frac{\mathbf{e}(\mathbf{e}k^2 \alpha)^{\ell-1}}{\ell n^3 (\mathbf{e}k^2 \alpha)^\ell} = \frac{1}{n^3 k^2 \alpha \ell}$$

Finally, using a union bound again, we obtain that

$$\Pr\left[\exists 2 \le \ell \le m \text{ and clause } c \text{ such that } Z_{\ell,c} \ge n^3 (ek^2 \alpha)^\ell\right]$$
$$\le \sum_{\ell=2}^m \frac{m}{n^3 k^2 \alpha \ell} = \frac{1}{k^2 n^2} \sum_{\ell=2}^m \frac{1}{\ell} \le \frac{\log m}{k^2 n^2} = o(1/n).$$

Next, we show that H_{Φ} satisfies Property 3.10 and Property 3.11 with probability 1 - o(1/n) in Lemma A.12 and Lemma A.14. To prove these two lemmas, we also need the following properties for random formulas and for Algorithm 1.

In a hypergraph $H = (V, \mathcal{E})$, let $\Gamma_H(V')$ denote the set of neighbors of vertices in V', and let $\Gamma_H^+(V') = V' \cup \Gamma_H(V')$. Then we have the following technical propositions.

Proposition A.7 ([HWY23, Proposition 3.6]). Let $\Phi = (k, n, m)$ be a random k-SAT formula with $k \ge 30$ and density α . Then with probability 1 - o(1/n), we have

$$\left|\Gamma_{H_{\Phi}}^{+}(V')\right| \le 3k^{4}\alpha \max\{|V'|, k \log n\}$$

for every connected subset V' of vertices in H_{Φ} .

Proof. Define $\mathcal{E}' = \{e \in \mathcal{E} \mid vbl(e) \cap V' \neq \emptyset\}$. It suffices to bound $|\mathcal{E}'| \leq 3k^3 \alpha \max\{|V'|, k \log n\}$ since $|\Gamma^+_{H_{\Phi}}(V')| \leq k|\mathcal{E}'|$.

We turn our focus to the case when $|V'| \ge \lfloor k \log n \rfloor$. Since $H_{\Phi}(V')$ is connected, there exists a $\mathcal{E}'' \subset \mathcal{E}'$ such that V' is connected using hyperedges in \mathcal{E}'' and $|V'|/k \le \mathcal{E}'' \le |V'|$. We get that

$$|\mathcal{E}''| \ge \log n - 1.$$

Now, define $\tilde{\mathcal{E}} = \mathcal{E}' \setminus \mathcal{E}''$. Since $k^3 \alpha \ge 1$, we will focus on bounding $|\mathcal{E}'| \le 2k^3 \alpha |V'| + |V'|$. Using the fact that $|\mathcal{E}'| \le |\tilde{\mathcal{E}}| + |\mathcal{E}''|$ and $|\mathcal{E}''| \le |V'|$, we focus our attention on bounding $|\tilde{\mathcal{E}}| \le 2k^3 \alpha |V'|$. Consider any fixed $\mathcal{E}'', V', \tilde{\mathcal{E}}$ which satisfy:

- $|\mathcal{E}''| \ge \log n 1$, $|V'| \ge |\mathcal{E}''|$, $|\tilde{\mathcal{E}}| \ge 2k^3 \alpha |V'|$, and $\tilde{\mathcal{E}} \cap \mathcal{E}'' = \emptyset$;
- Further, $G_{\Phi}(\mathcal{E}'')$ is connected, $V' \subset \bigcup_{e \in \mathcal{E}''} \operatorname{vbl}(\mathcal{E}'')$ and $\operatorname{vbl}(e) \cap V' \neq \emptyset$ for all $e \in \mathcal{E}''$.

Define $r_1 = |\mathcal{E}''|$, $r_2 = |V'|$, and $r_3 = |\tilde{\mathcal{E}}|$. Based on these observations, we define the events:

- $\mathcal{A}(\mathcal{E}'', V', \tilde{\mathcal{E}})$ to be the event when the above conditions are satisfied with $(\mathcal{E}'', V', \tilde{\mathcal{E}})$;
- $\mathcal{A}(\mathcal{E}'')$ to be event that $G_{\Phi}[\mathcal{E}'']$ is connected;
- $\mathcal{A}(V', \tilde{\mathcal{E}})$ to be the event that $vbl(e) \cap V' \neq \emptyset$ holds for all $e \in \tilde{\mathcal{E}}$.

Now, using Proposition A.5, we get that

$$\Pr\left[\mathcal{A}(\mathcal{E}'']\right) \leq r_1^{r_1-2} \cdot \left(\frac{k^2}{n}\right)^{r_1-1}$$

Observe that since $\mathcal{E}'' \cap \tilde{\mathcal{E}} = \emptyset$, $\mathcal{A}(\mathcal{E}'')$ and $\mathcal{A}(V', \tilde{\mathcal{E}})$ are independent. Then,

$$\Pr\left[\mathcal{A}(V',\tilde{\mathcal{E}})|\mathcal{A}(\mathcal{E}'')\right] = \Pr\left[\mathcal{A}(V',\tilde{\mathcal{E}})\right] \le \left(k \cdot \frac{r_2}{n}\right)^{r_3}$$

Therefore,

$$\mathbf{Pr}\left[\mathcal{A}(\mathcal{E}'',V',\tilde{\mathcal{E}})\right] \leq \mathbf{Pr}\left[\mathcal{A}(V',\tilde{\mathcal{E}})\right] \cdot \mathbf{Pr}\left[\mathcal{A}(\mathcal{E}'')\right] \leq r_1^{r_1-2} \cdot \left(\frac{k^2}{n}\right)^{r_1-1} \left(k \cdot \frac{r_2}{n}\right)^{r_3}.$$

We now use union bound over all possible (valid) sizes of $\mathcal{E}'', V', \tilde{\mathcal{E}}$ to upper bound the probability that under the conditions mentioned above, $r_3 \ge 2k^3 \alpha r_2$. We will show this is upper bounded by o(1/n), which then implies our result. By the union bound,

$$\mathbf{Pr}\left[\exists \operatorname{such} \mathcal{A}(\mathcal{E}'', V', \tilde{\mathcal{E}})\right] \leq \sum_{r_1 \geq \log n-1} \sum_{r_2 \geq r_1} \sum_{r_3 \geq 2k^3 \alpha r_2} \binom{m}{r_1} \binom{kr_1}{r_2} \binom{m}{r_3} r_1^{r_1-2} \cdot \left(\frac{k^2}{n}\right)^{r_1-1} \left(k \cdot \frac{r_2}{n}\right)^{r_3}.$$

To simplify the above expression, we make following assumptions: $\frac{ek^2\alpha}{(2k^2/e)^{k^3\alpha}} \leq \frac{1}{8}$ and $\frac{ek}{(2k^2/e)^{k^3\alpha}} \leq \frac{1}{8}$. These assumptions are satisfied under $k^3\alpha \geq 1$ and $k \geq 30$. Then, we get that

$$\Pr\left[\exists \operatorname{such} \mathcal{A}(\mathcal{E}'', V', \tilde{\mathcal{E}})\right] \leq \frac{4n}{n^2 (\log n - 1)^2} = o(1/n).$$

For the case when $|V'| < \lfloor k \log n \rfloor$, we can consider another connected set of vertices $V'' \supset V'$ such that $|V''| = \lfloor k \log n \rfloor$. Applying the previous argument on V'', the claim follows. \Box

Using Proposition A.7, We can also bound the fraction of high-degree variables in any connected set.

Proposition A.8. Assume $k \ge 2$, $q \ge 2$, and $\alpha \le q^k$ are constants. Let p_1 be a parameter satisfying $6k^5 \le p_1 \le e^{k-2}\alpha$. Then with probability 1 - o(1/n) over the random formula Φ , the following holds for $H_{\Phi} = (V, \mathcal{E})$: Let $V' \subseteq V$ be connected in H_{Φ} and $|V'| \ge \log n$. Then

$$|\{v \in V' \mid \deg(v) \ge p_1 \alpha\}| \le 6k^5 |V'| / p_1$$

Proof. We prove this proposition by showing that if HD(V') is too large then V' has too many neighbors, which contradicts to Proposition A.7.

We consider the number of edges that contain high-degree vertices in V'. Let

$$\mathcal{E}' = \{ e \in \mathcal{E} \mid e \cap \operatorname{HD}(V') \neq \emptyset \}, \text{ and } U = \bigcup_{e \in \mathcal{E}'} e.$$

By setting $\eta = 1/2$ and $\rho = e^{-2}/(k\alpha)$ in Lemma A.3, we obtain that any $\ell \leq \rho m$ edges contain at least $k\ell/2$ distinct vertices with probability 1 - o(1/n). Thus, it gives that $|U| \geq k \min\{|\mathcal{E}'|, \rho m\}/2$. Note that $U = \Gamma_{H_{\Phi}}^{+}(\operatorname{HD}(V')) \subseteq \Gamma_{H_{\Phi}}^{+}(V')$. So we have $\left|\Gamma_{H_{\Phi}}^{+}(V')\right| \geq k \min\{|\mathcal{E}'|, \rho m\}/2$. Now we bound the size of \mathcal{E}' . By a double counting on the size of $\{(v, e) \in \operatorname{HD}(V') \times \mathcal{E}' \mid v \in e\}$,

Now we bound the size of \mathcal{E}' . By a double counting on the size of $\{(v, e) \in HD(V') \times \mathcal{E}' \mid v \in e\}$, we have $k |\mathcal{E}'| \ge p_1 \alpha |HD(V')|$, and thus

(21)
$$|\operatorname{HD}(V')| \leq \frac{k}{p_1 \alpha} |\mathcal{E}'| .$$

If $|\mathcal{E}'| \leq \rho m$, then by Proposition A.7, it follows that

$$\frac{k\left|\mathcal{E}'\right|}{2} \le \left|\Gamma_{H_{\Phi}}^{+}(V')\right| \le 3k^{4}\alpha \max\{\left|V'\right|, k\log n\} \le 3k^{5}\alpha \left|V'\right|.$$

Combining with (21), it yields that $|\mathcal{E}'| \leq 6k^4 \alpha |V'|$, and thus

$$|\mathrm{HD}(V')| \le \frac{6k^5}{p_1} |V'|$$

If $|\mathcal{E}'| > \rho m$, using Proposition A.7 again, we obtain

$$\frac{k\rho m}{2} \le \left|\Gamma_{H_{\Phi}}^{+}(V')\right| \le 3k^{4}\alpha \max\{\left|V'\right|, k\log n\} \le 3k^{5}\alpha \left|V'\right|,$$

which gives

$$|V'| \ge \frac{k\rho m}{6k^5\alpha} = \frac{n}{6\mathrm{e}^2k^5\alpha} \,.$$

On the other hand, it is clear that $|\text{HD}(V')| \leq |\text{HD}(V)| \leq e^{-k} \alpha^{-2} n$ by Proposition A.2. So we obtain that

$$|\text{HD}(V')| \le \frac{6e^2k^5\alpha}{e^k\alpha^2} |V'| \le \frac{6k^5}{p_1} |V'|$$
.

The following result is adapted from [COF14, Lemma 2.4] and [HWY23, Lemma A.2], but uses tighter parameters.

Proposition A.9. Assume $k \ge 12$, $q \ge 2$ and $\alpha \le q^k$ are constants. Then with probability 1 - o(1/n)over the random formula Φ , the following holds for $H_{\Phi} = (V, \mathcal{E})$: Fix an arbitrary $\mathcal{E}' \subseteq \mathcal{E}$ of size $|\mathcal{E}'| \leq 4^{-k} \alpha^{-1/2} n$. Let $e_{i_1}, \ldots, e_{i_\ell} \in \mathcal{E} \setminus \mathcal{E}'$ be hyperedges of distinct indices. For each $s \in [\ell]$, define $V_s = (\bigcup_{e \in \mathcal{E}'} e) \cup (\bigcup_{j=1}^{s-1} e_{i_j})$. If $|e_{i_s} \cap V_s| \geq 6$ holds for all $s \in [\ell]$, then $\ell \leq |\mathcal{E}'|$.

Proof. We prove the statement by contradiction. Hence, assume that \mathcal{E}' and $e_{i_1}, ..., e_{i_\ell}$ violate the statement for $\ell > |\mathcal{E}'|$. We can discard additional clauses from \mathcal{E}' and in fact, assume that $\ell = |\mathcal{E}'| + 1$ for which the statement is violated. Further, observe that we can assume that $|\mathcal{E}'| \leq m - \ell$. Define $\varepsilon > 0$ such that $\ell = \lfloor \varepsilon n \rfloor + 1$. Since $|\mathcal{E}'| \ge 1$, we have that $\varepsilon \ge \frac{1}{n}$. From the bound on $|\mathcal{E}'|$, we get that $\varepsilon \leq 4^{-k}\alpha^{-1/2} + \frac{1}{n}$. Define $Y := \bigcup_{j=1}^{j=\ell} \operatorname{vbl}(e_{i_j}) \setminus \bigcup_{e \in \mathcal{E}'} \operatorname{vbl}(e)$. Then, the following are true:

- $|Y| = \sum_{r=1}^{r=\ell} |\operatorname{vbl}(e_{i_r})| |\operatorname{vbl}(e_{i_r}) \cap V_r| \le (k-6)\ell;$ There is a $\tilde{\mathcal{E}} \subset \mathcal{E} \setminus \mathcal{E}'$ such that $|\tilde{\mathcal{E}}| = \ell$ and $\operatorname{vbl}(\tilde{e}) \subseteq Y \cup \bigcup_{e \in \mathcal{E}'} \operatorname{vbl}(e)$ for all $e \in \tilde{\mathcal{E}}$. For this, we can just take $\tilde{\mathcal{E}} = \{e_{i_1}, ..., e_{i_\ell}\}.$

Now, consider fixed $\mathcal{E}', Y, \tilde{\mathcal{E}}$ such that $|Y| = t \le (k-6)\ell$, $|\tilde{\mathcal{E}}| = \ell$ and $|\mathcal{E}'| = \min\{m-\ell, \ell-1\}$. Define $\mathcal{A}(\mathcal{E}', Y, \tilde{\mathcal{E}})$ to be the event that $vbl(\tilde{\mathcal{E}}) \subset Y \cup vbl(\mathcal{E}')$. We have that

$$\mathbf{Pr}\left[\mathcal{A}(\mathcal{E}',Y,\tilde{\mathcal{E}})\right] \le \left(\frac{k|\mathcal{E}'|+|Y|}{n}\right)^{k\tilde{\mathcal{E}}} \le \left(\frac{k(\ell-1)+(k-6)\ell}{n}\right)^{k\ell} \le (4k\varepsilon)^{k\ell}.$$

The last inequality follows from the fact that $\ell \leq \varepsilon n + 1 \leq 2\varepsilon n$. Now, by union bound over all choices of $(\mathcal{E}', Y, \mathcal{E})$, we get that

$$\mathbf{Pr}\left[\exists \text{ such an } (\mathcal{E}', Y, \tilde{\mathcal{E}})\right] \leq \sum_{t=1}^{t=k-6} \binom{m}{\ell-1} \cdot \binom{m}{\ell} \cdot \binom{n}{t} (4k\varepsilon)^{k\ell}.$$

We can upper bound both $\binom{m}{\ell}$ and $\binom{m}{\ell-1}$ by $(\frac{em}{\ell-1})^{\ell}$. Further, $(k-6)\ell \leq (k-6)(\varepsilon n+1) \leq 2k\varepsilon n \leq n/2$ if we assume $\varepsilon \leq 1/(4k)$. But we have that $\varepsilon \leq 4^{-k}\alpha^{-1/2} + \frac{1}{n}$ and we can use that $k \geq 12$. Then, $\binom{n}{t} \leq \left(\frac{\mathrm{e}n}{(k-6)\ell}\right)^{(k-6)\ell}$. Now,

$$\begin{aligned} \mathbf{Pr}\left[\exists \text{ such an } (\mathcal{E}', Y, \tilde{\mathcal{E}})\right] &\leq n \cdot \left(\frac{\mathrm{e}\alpha n}{\ell - 1}\right)^{2\ell} \cdot \left(\frac{\mathrm{e}n}{(k - 6)\ell}\right)^{(k - 6)\ell} \cdot (4k\varepsilon\ell)^{k\ell} \\ &\leq n \cdot \left(\frac{\mathrm{e}^{k - 4} \cdot \alpha^2 \cdot 2^{2k} \cdot n^{k - 4} \cdot \varepsilon^k \cdot k^k}{\ell^{k - 6} \cdot (\ell - 1)^2 (k - 6)^{k - 6}}\right)^{\ell}. \end{aligned}$$

Now, to further evaluate the R.H.S., we use the fact that $\frac{k^k}{(k-6)^{k-6}} \le (ek)^6$. Also, $\ell \le \varepsilon n$ and $\ell - 1 \ge \varepsilon n/2$. Then, we get that

$$\mathbf{Pr}\left[\exists \text{ such an } (\mathcal{E}', Y, \tilde{\mathcal{E}})\right] \le n \cdot \left(e^{k+2}2^{2k} \cdot \alpha^2 \cdot \varepsilon^4 \cdot k^6\right)^{\ell} := \tilde{p}.$$

Observe that since $\varepsilon \leq 4^{-k}\alpha^{-1/2} + \frac{1}{n}$, we get that $e^{k+2}2^{2k} \cdot \alpha^2 \cdot \varepsilon^4 \cdot k^6 \leq \frac{1}{2}$ since $k \geq 12$. Now, based on whether $\varepsilon n \geq 5 \log n$ or $\varepsilon n < 5 \log n$, we make two cases:

- If $\varepsilon n \ge 5 \log n$, we get that $\tilde{p} \le n \cdot (\frac{1}{2})^{\varepsilon n} = o(1/n^3)$;
- If $\varepsilon n < 5 \log n$, we assume $n \ge 2^{\Omega(k)}$ (for an appropriately large constant), which implies that $e^{k+2}2^{2k} \cdot \alpha^2 \cdot \varepsilon^4 \cdot k^6 = o(1/n^3)$ and we also have that $\ell \ge 2$, which again implies that $\tilde{p} = o(1/n^3)$.

Observe that we have shown the above probability bound for \mathcal{E}' of a specific size, which, in our case was $\min\{\ell-1, m-\ell\}$. Then, by union bound over all possible sizes of \mathcal{E}' , we get that $\Pr\left[\exists \text{ no } (\mathcal{E}', Y, \tilde{\mathcal{E}})\right] \ge 1 - o(1/n)$.

Using Proposition A.9, we can bound the size of bad vertices and bad hyperedges in any connected sets. Given a hypergraph $H = (V, \mathcal{E})$, we now simplify $V_{\text{bad}}(V)$ to V_{bad} and $\mathcal{E}_{\text{bad}}(V)$ to \mathcal{E}_{bad} . We first present the following facts.

Fact A.10. For any $V' \subseteq V$, it holds that $V_{\text{bad}}(V') \subseteq V_{\text{bad}}$ and $\mathcal{E}_{\text{bad}}(V') \subseteq \mathcal{E}_{\text{bad}}$.

Fact A.11 ([GGGY21, Lemma 8.9]). For any $V' \subseteq V_{\text{bad}}$ such that V' is a connected component in the induced subgraph $H[V_{\text{bad}}]$, it holds that $V_{\text{bad}}(V') = V'$.

If Proposition A.9 holds for H_{Φ} , we can prove that Property 3.10 is also satisfied, which states that the number of bad variables is bounded by the number of high-degree variables.

Lemma A.12. For any fixed k and α , if $\eta k \ge 4$, $e(\rho k \alpha)^{\eta} \le 1$, $\varepsilon_1 > \eta$, and $6k^5 \le p_1 \le e^{k-2}\alpha$, then with probability 1 - o(1/n) over the choice of random formula Φ , it holds for $H_{\Phi} = (V, \mathcal{E})$ that:

- (1) for any $V' \subseteq V$, we have $|V_{bad}(V')| \le 2 |HD(V')| / (\varepsilon_1 \eta)$, in particular, H_{Φ} satisfies Property 3.10 if $\varepsilon_1 = 2\eta$;
- (2) for any $V' \subseteq V_{\text{bad}}$ such that V' consists of some connected components in $H_{\Phi}[V_{\text{bad}}]$, we have $|V'| \leq 2 |\text{HD}(V')| / (\varepsilon_1 \eta)$.

Proof. We only need to prove item (1). Item (2) is a direct corollary of item (1) and Fact A.11.

By Proposition A.2, we have $|HD(V')| \le |HD(V)| \le e^{-k} \alpha^{-2} n$. Let

$$\mathcal{E}' = \{ e \in \mathcal{E} \mid |e \cap \mathrm{HD}(V')| \ge \varepsilon_1 k \}.$$

By setting $b = \varepsilon_1$ in Lemma A.3 and Corollary A.4, it follows that

$$|\mathrm{HD}(V')| \ge (\varepsilon_1 - \eta)k |\mathcal{E}'|$$

In particular, $|\mathcal{E}'| \leq |\text{HD}(V')| \leq e^{-k} \alpha^{-2} n$. Observe that, in Algorithm 1, if there exist more than one hyperedge that can be added into $\mathcal{E}_{\text{bad}}(V')$, we can add them in an arbitrary order without changing the output of Algorithm 1. So we start by adding all hyperedges in \mathcal{E}' into $\mathcal{E}_{\text{bad}}(V')$ first. After that, each hyperedge newly added to $\mathcal{E}_{\text{bad}}(V')$ intersects at least $\varepsilon_1 k$ vertices with existing hyperedge in $\mathcal{E}_{\text{bad}}(V')$. Hence, applying Proposition A.9, we obtain that $|\mathcal{E}_{\text{bad}}(V') \setminus \mathcal{E}'| \leq |\mathcal{E}'|$, which implies that

$$|\mathcal{E}_{\text{bad}}(V')| \le 2 |\mathcal{E}'| \le \frac{2}{(\varepsilon_1 - \eta)k} |\text{HD}(V')|$$

and thus

$$|V_{\text{bad}}(V')| \le k |\mathcal{E}_{\text{bad}}(V')| \le \frac{2}{\varepsilon_1 - \eta} |\text{HD}(V')| .$$

Now we can prove Property 3.11. The following two results are adapted from [HWY23, Lemma 4.4 & Corollary 4.2] using our parameters.

Proposition A.13. For any fixed k and α , assume η , ρ , p_1 , ε_1 are parameters satisfying $\eta k \geq 4$, $e(\rho k \alpha)^{\eta} \leq 1$, $\varepsilon_1 \geq \eta + 1/k$ and $6k^5 \leq p_1 \leq e^{k-2}\alpha$. Then with probability 1 - o(1/n) over the choice of random formula Φ , for any $V' \subseteq V$ of size $|V'| \geq \log n$ connected in $H_{\Phi} = (V, \mathcal{E})$, it holds that $|V' \cap V_{\text{bad}}| \leq \frac{12k^5}{(\varepsilon_1 - \eta)p_1} |V'|$.

Proof. Let V_1, V_2, \ldots, V_ℓ be connected components in $H_{\Phi}[V_{\text{bad}}]$ that intersects V', and let

$$\tilde{V} = V' \cup V_1 \cup V_2 \cup \cdots \cup V_\ell.$$

Note that \tilde{V} is connected in H_{Φ} , and $HD(\tilde{V}) = HD(V_1) \cup HD(V_2) \cup \cdots \cup HD(V_{\ell})$.

By Lemma A.12, we have $|V_i| \leq 2 |\text{HD}(V_i)| / (\varepsilon_1 - \eta)$. By Proposition A.8, we have $|\text{HD}(\tilde{V})| \leq 6k^5 |\tilde{V}| / p_1$. Thus, it follows that

$$\left|\tilde{V} \cap V_{\text{bad}}\right| = \sum_{i=1}^{\ell} |V_i| \le \frac{2}{\varepsilon_1 - \eta} \sum_{i=1}^{\ell} |\text{HD}(V_i)| \le \frac{2}{\varepsilon_1 - \eta} \cdot \frac{6k^5 \left|\tilde{V}\right|}{p_1}.$$

Since $\tilde{V} \setminus V' \subseteq V_{\text{bad}}$, we conclude that

$$\frac{|V' \cap V_{\text{bad}}|}{|V'|} \le \frac{|V' \cap V_{\text{bad}}| + \left|\tilde{V} \setminus V'\right|}{|V'| + \left|\tilde{V}\right|} = \frac{\left|\tilde{V} \cap V_{\text{bad}}\right|}{\left|\tilde{V}\right|} \le \frac{12k^5}{(\varepsilon_1 - \eta)p_1} \,.$$

As a corollary, we obtain the proof of Property 3.11.

Lemma A.14. For any fixed k and α , assume η , ρ , p_1 , ε_1 are parameters satisfying $\eta k \ge 4$, $e(\rho k \alpha)^{\eta} \le 1$, $\varepsilon_1 \ge \eta + 1/k$ and $6k^5 \le p_1 \le e^{k-2}\alpha$. Then with probability 1 - o(1/n) over the choice of random formula Φ , for any $\mathcal{E}' \subseteq \mathcal{E}$ of size $|\mathcal{E}'| \ge \log n$ connected in the line graph of $H_{\Phi} = (V, \mathcal{E})$ (namely, connected in G_{Φ}), it holds that $|\mathcal{E}' \cap \mathcal{E}_{bad}| \le \frac{12k^5}{(1-\eta)(\varepsilon_1-\eta)p_1} |\mathcal{E}'|$. In particular, H_{Φ} satisfies Property 3.11 if $\varepsilon_1 = 2\eta$ and $\varepsilon_2 = \frac{12k^5}{(1-\eta)\eta p_1}$.

Proof. Let $V' = \bigcup_{e \in \mathcal{E}'} e$. Clearly, V' is connected in H_{Φ} , and $|V'| \leq k |\mathcal{E}'|$.

We first assume that $|\mathcal{E}'| \leq \rho |\mathcal{E}|$. Then Lemma A.3 applies. It implies that $|V'| \geq (1-\eta)k |\mathcal{E}'| \geq \log n$, and thus Proposition A.13 applies. Therefore, we obtain

$$|V' \cap V_{\text{bad}}| \le \frac{12k^5}{(\varepsilon_1 - \eta)p_1} |V'| \le \frac{12k^6}{(\varepsilon_1 - \eta)p_1} |\mathcal{E}'| .$$

Note that, each hyperedge in $\mathcal{E}' \cap \mathcal{E}_{bad}$ is a subset of $V' \cap V_{bad}$. Applying Lemma A.3 again, we have

$$|\mathcal{E}' \cap \mathcal{E}_{\text{bad}}| \le \frac{|V' \cap V_{\text{bad}}|}{(1-\eta)k} \le \frac{12k^5}{(1-\eta)(\varepsilon_1 - \eta)p_1} |\mathcal{E}'|$$

Now we consider the case where $|\mathcal{E}'| > \rho |\mathcal{E}|$. Again, note that each hyperedge in $\mathcal{E}' \cap \mathcal{E}_{bad}$ is a subset of V_{bad} , and $|V_{bad}| \le 2e^{-k}\alpha^{-2} |V|/(\varepsilon_1 - \eta) < \rho |\mathcal{E}|/((1 - \eta)k)$ by Proposition A.2 and Lemma A.12 if we set $\eta \ge 4/k$ and $e(\rho k \alpha)^{\eta} = 1$. So Lemma A.3 applies to V_{bad} , and we conclude that

$$|\mathcal{E}' \cap \mathcal{E}_{\text{bad}}| \le \frac{|V_{\text{bad}}|}{(1-\eta)k} \le \frac{2|\mathcal{E}|}{\mathrm{e}^k k \alpha^3 (1-\eta)(\varepsilon_1 - \eta)} < \frac{12k^5}{(1-\eta)(\varepsilon_1 - \eta)p_1} |\mathcal{E}'| .$$

Finally, it is a direct corollary of Property 3.11 that H_{Φ} has no connected components of size $\ell \geq \log n$ in the line graph induced by bad hyperedges.