

Local Gibbs Sampling Beyond Local Uniformity

Chunyang Wang

(Nanjing University→National Institute of Informatics)

Joint work with:

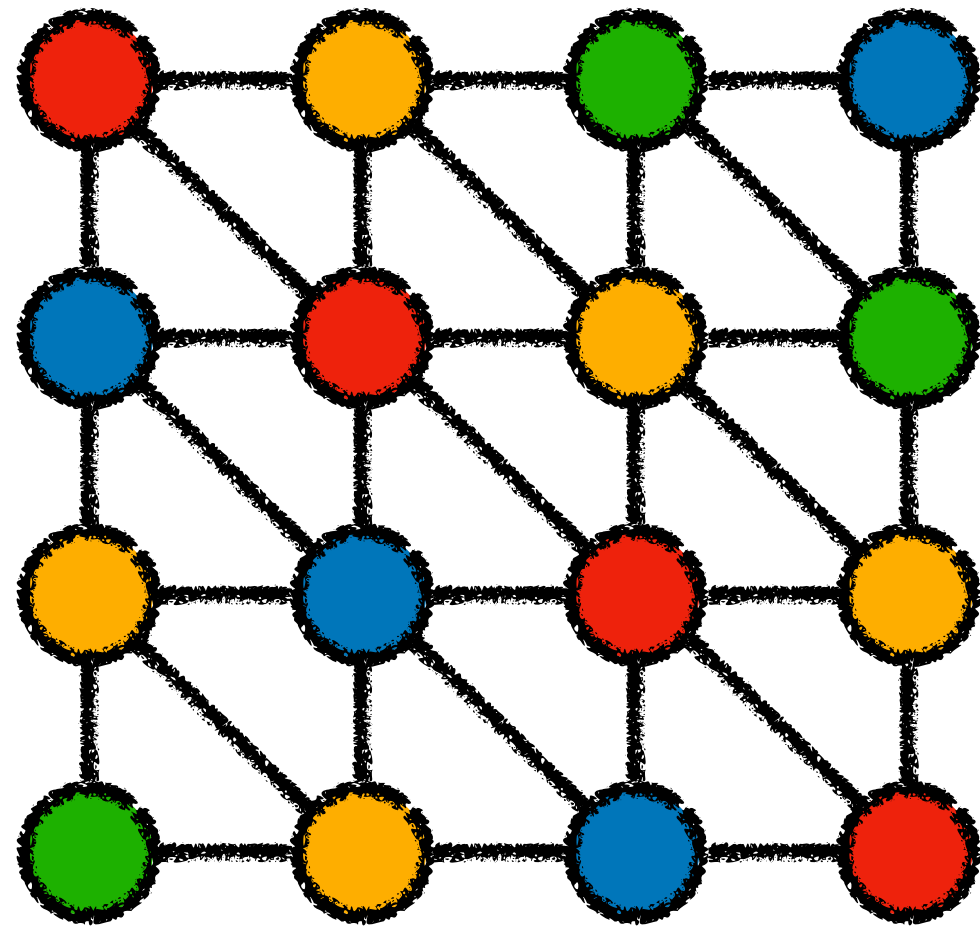


Hongyang Liu
(Nanjing University)



Yitong Yin
(Nanjing University)

(Multi-) Spin Systems



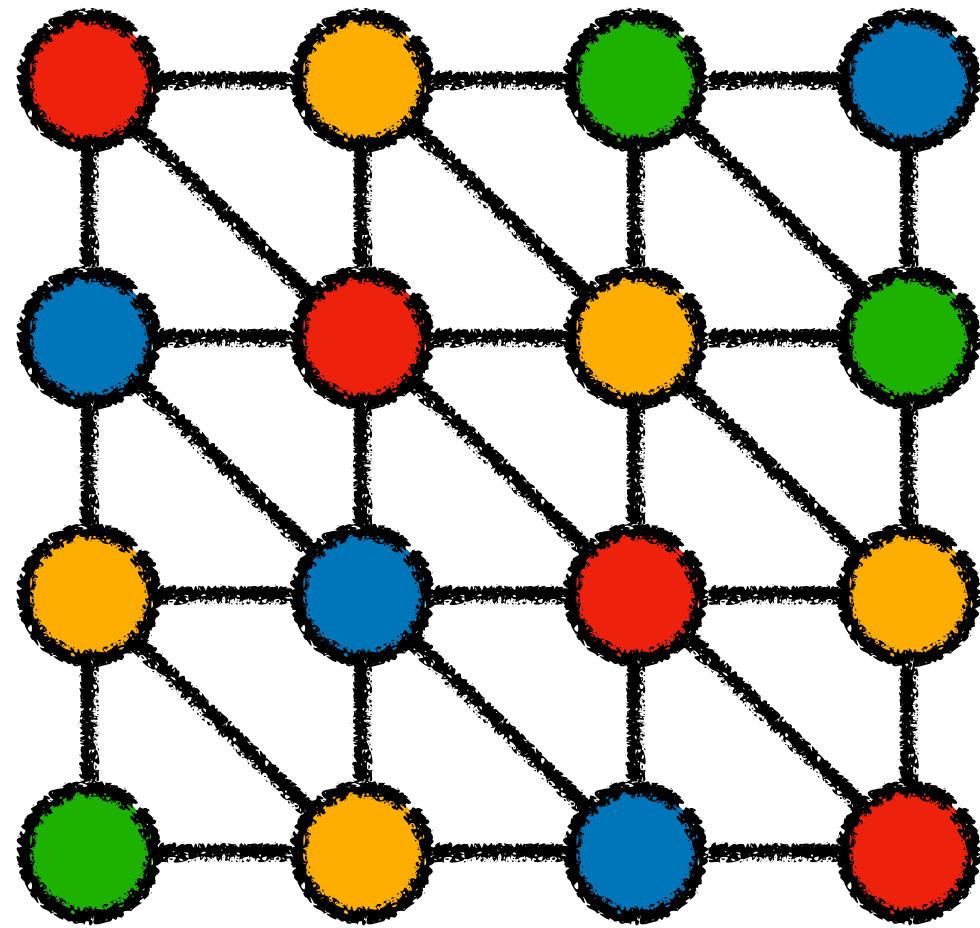
graph $G = (V, E)$ $q \geq 2$ states

configuration $\sigma \in [q]^V$

external fields $\lambda_v \in \mathbb{R}_{\geq 0}^q$ for each $v \in V$

interaction matrix $A_e \in \mathbb{R}_{\geq 0}^{q \times q}$ for each $e \in E$

(Multi-) Spin Systems



graph $G = (V, E)$ $q \geq 2$ states

configuration $\sigma \in [q]^V$

external fields $\lambda_v \in \mathbb{R}_{\geq 0}^q$ for each $v \in V$

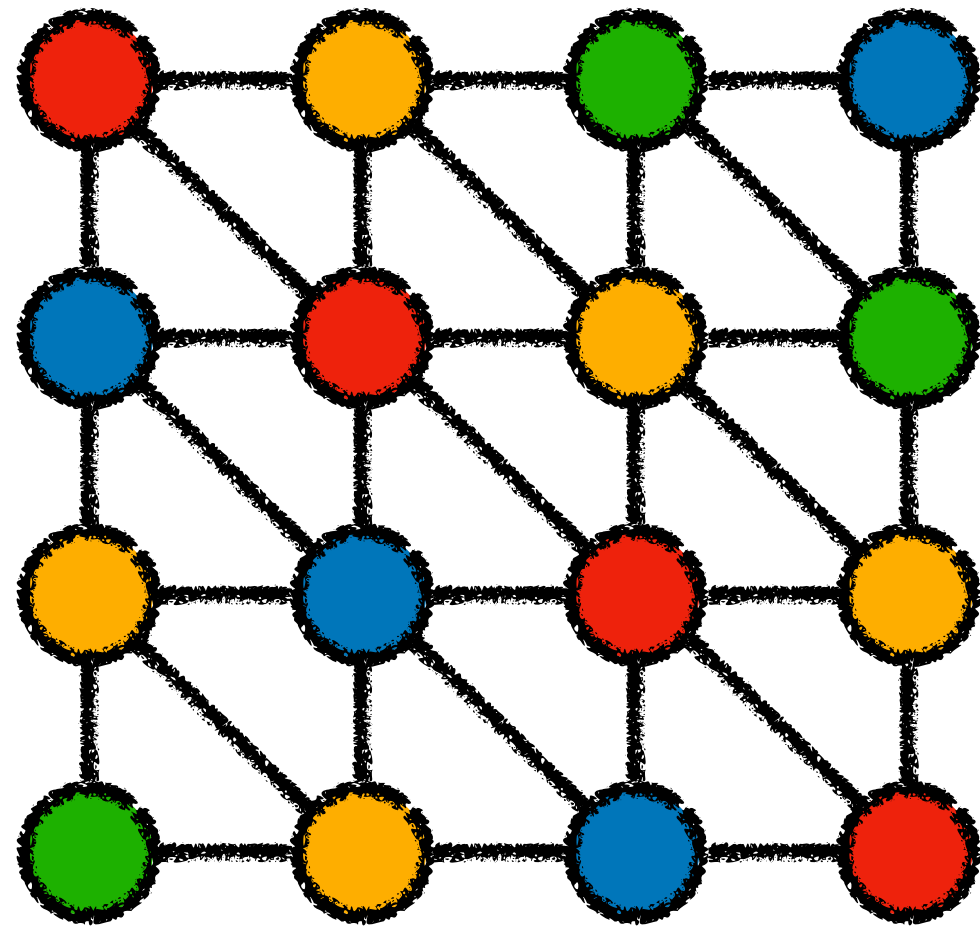
interaction matrix $A_e \in \mathbb{R}_{\geq 0}^{q \times q}$ for each $e \in E$

$$\text{weight: } w(\sigma) = \prod_{v \in V} \lambda_v(\sigma(v)) \prod_{(u,v) \in E} A_e(\sigma(u), \sigma(v))$$

$$\text{partition function: } Z = \sum_{\sigma \in [q]^V} w(\sigma)$$

$$\text{Gibbs distribution: } \mu(\sigma) = \frac{w(\sigma)}{Z}$$

(Multi-) Spin Systems



graph $G = (V, E)$ $q \geq 2$ states

configuration $\sigma \in [q]^V$

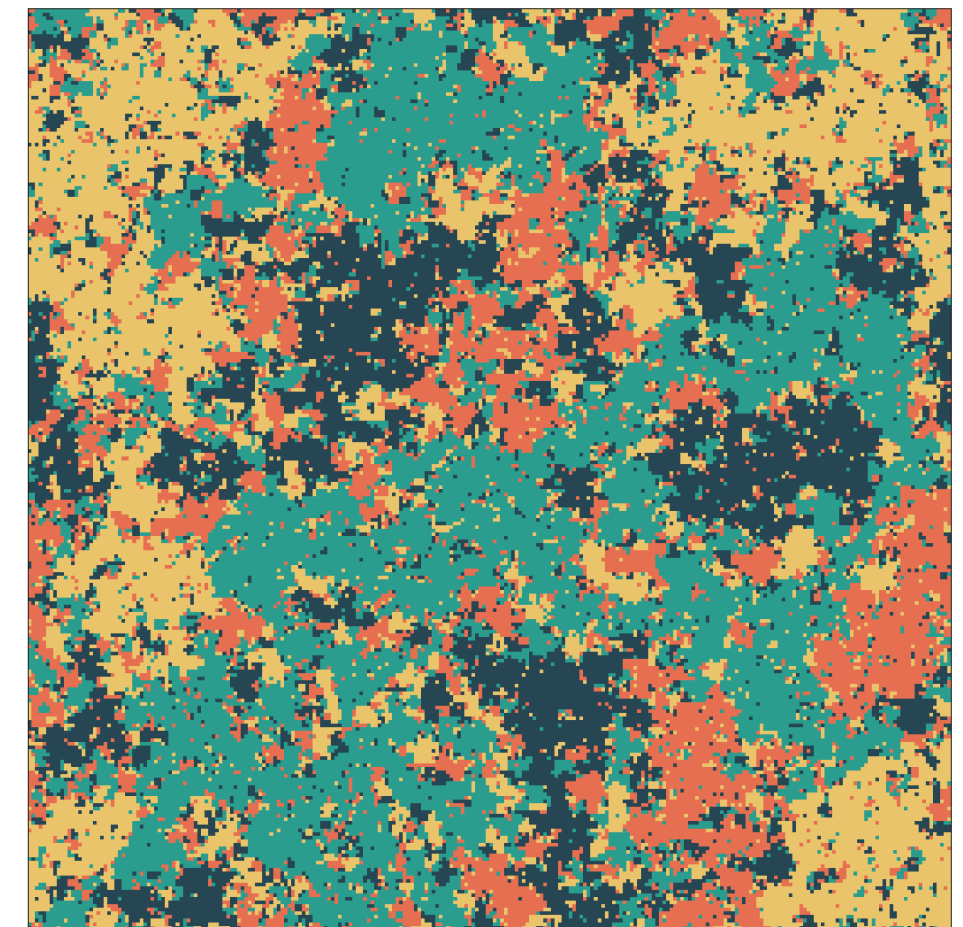
external fields $\lambda_v \in \mathbb{R}_{\geq 0}^q$ for each $v \in V$

interaction matrix $A_e \in \mathbb{R}_{\geq 0}^{q \times q}$ for each $e \in E$

$$\text{weight: } w(\sigma) = \prod_{v \in V} \lambda_v(\sigma(v)) \prod_{(u,v) \in E} A_e(\sigma(u), \sigma(v))$$

$$\text{partition function: } Z = \sum_{\sigma \in [q]^V} w(\sigma)$$

$$\text{Gibbs distribution: } \mu(\sigma) = \frac{w(\sigma)}{Z}$$



(Multi-) Spin Systems

Ising model

two states  

external fields  λ  1

interaction   β
  1

models weighted matchings

hardcore model

two states  

external fields  λ  1

interaction    1
 0

models weighted independent sets

Potts model

q states    

external fields     1

interaction     β
others 1

$\beta > 1$ (ferromagnetic): models clusterings

$\beta < 1$ (antiferromagnetic): models (soft) q -colorings

Sampling from Spin Systems

Sampling Problem

Input: a spin system $\mathcal{S} = (G = (V, E), (\lambda_v)_{v \in V}, (A_e)_{e \in E})$

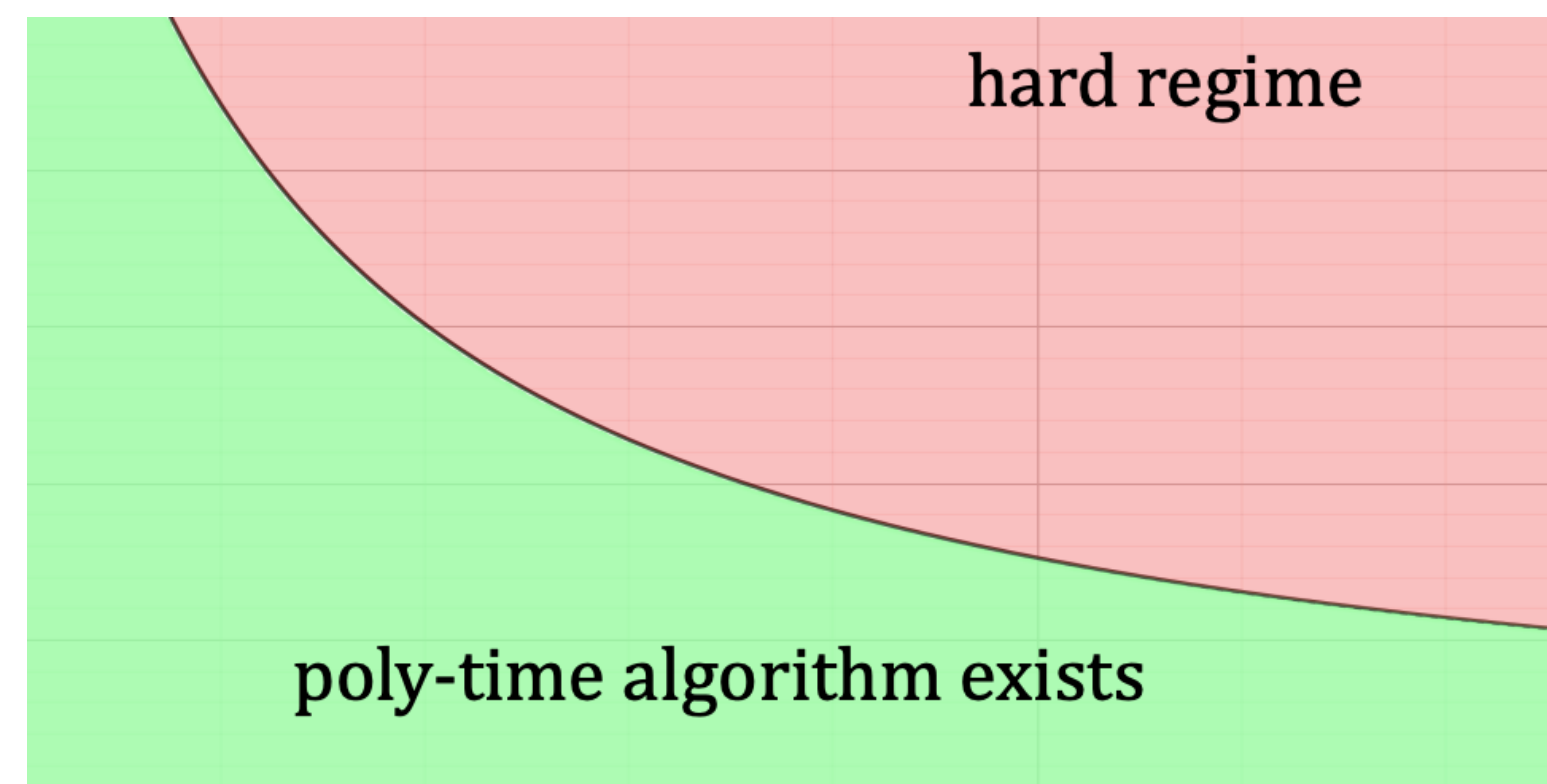
Output: a random configuration $X \sim \mu$

Sampling from Spin Systems

Sampling Problem

Input: a spin system $\mathcal{S} = (G = (V, E), (\lambda_v)_{v \in V}, (A_e)_{e \in E})$

Output: a random configuration $X \sim \mu$



Computational Phase Transition

computational complexity of the sampling problem

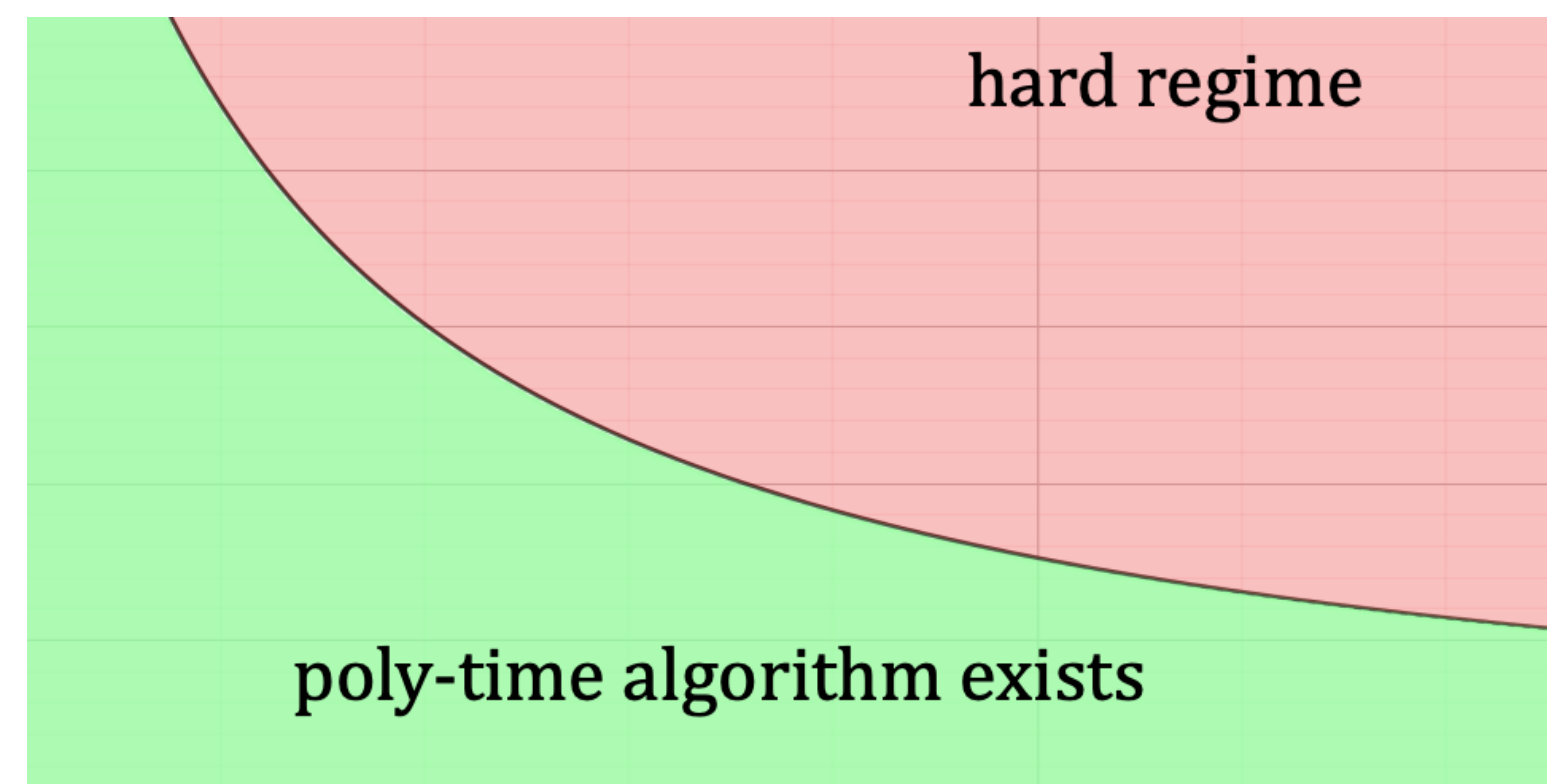
changes sharply around some parameters of $\mathcal{S}$

Sampling from Spin Systems

Sampling Problem

Input: a spin system $\mathcal{S} = (G = (V, E), (\lambda_v)_{v \in V}, (A_e)_{e \in E})$

Output: a random configuration $X \sim \mu$



Computational Phase Transition

For some typical models (Ising, hardcore):

outside the “uniqueness regime”: sampling is **NP-hard**! [SS ’14, GŠV ’16]

within the “uniqueness regime”: poly-time algorithms exist! [Weitz ’06, CLV ’20]

Sampling from Spin Systems

Glauber dynamics

Start with arbitrary configuration σ with $w(\sigma) > 0$;

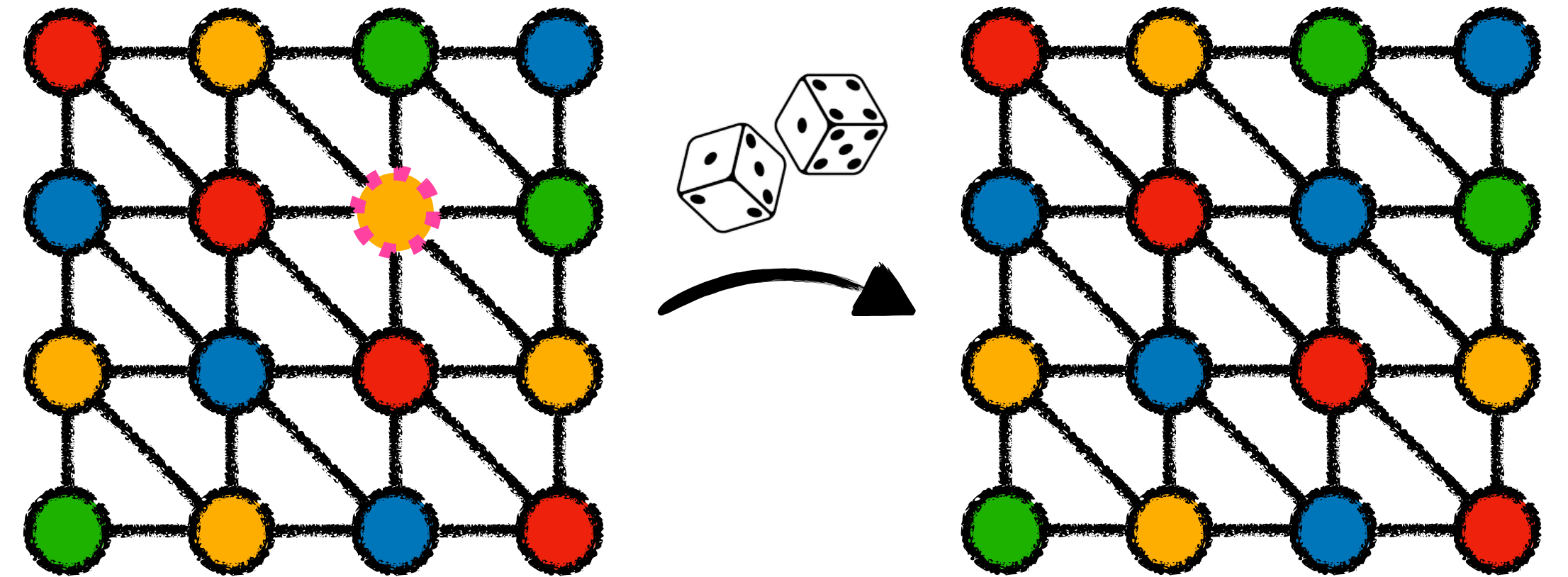
At each time $1 \leq t \leq T$:

Pick a vertex $v \in V$ uniformly at random;

Resample $\sigma_v \sim \mu_v \left(\cdot \mid \sigma_{V \setminus \{v\}} \right)$;

Return σ ;

converges to μ as $T \rightarrow \infty$!



Computational Phase Transition

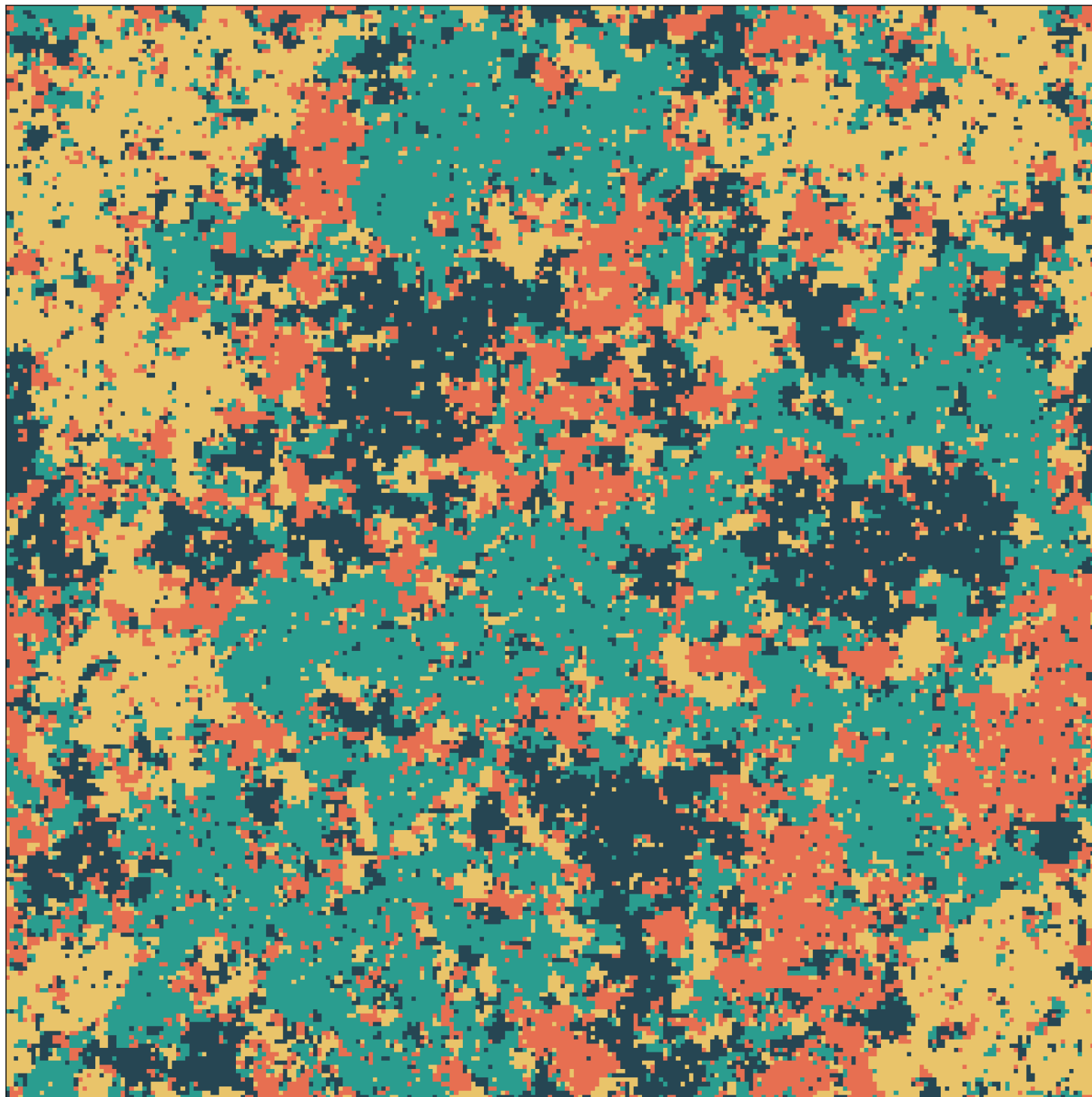
For some typical models (Ising, hardcore):

outside the “uniqueness regime”: sampling is **NP-hard**! [SS ’14, GŠV ’16]

within the “uniqueness regime”: **Glauber dynamics** mixes in near-linear time! [CLV ’20]

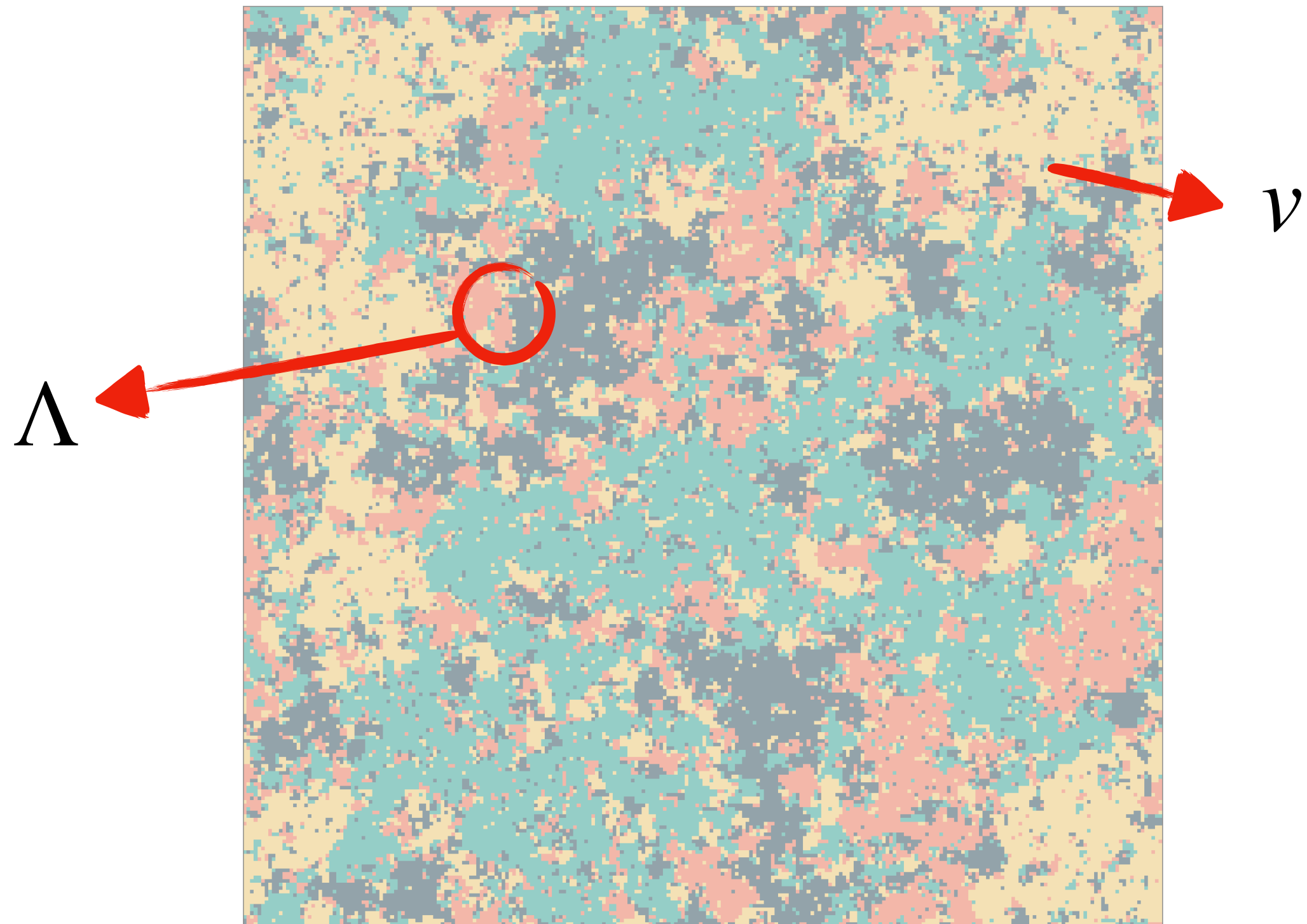
(Bayesian) probabilistic inference

Given a partial configuration σ_Λ over $\Lambda \subseteq V$ and a vertex $v \in V \setminus \Lambda$,
estimate the **conditional marginal probability distribution** $\mu_v^{\sigma_\Lambda}(\cdot)$



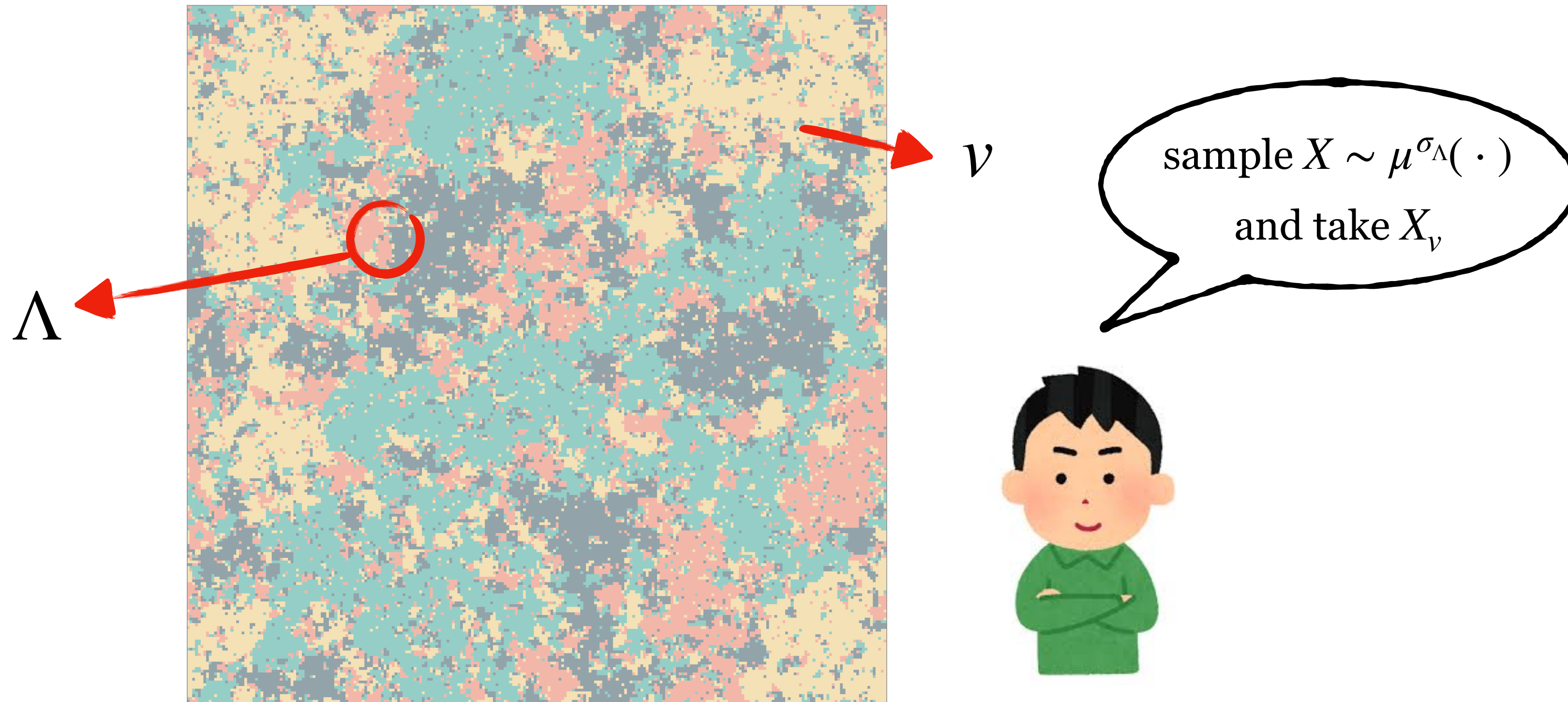
(Bayesian) probabilistic inference

Given a partial configuration σ_Λ over $\Lambda \subseteq V$ and a vertex $v \in V \setminus \Lambda$,
estimate the **conditional marginal probability distribution** $\mu_v^{\sigma_\Lambda}(\cdot)$



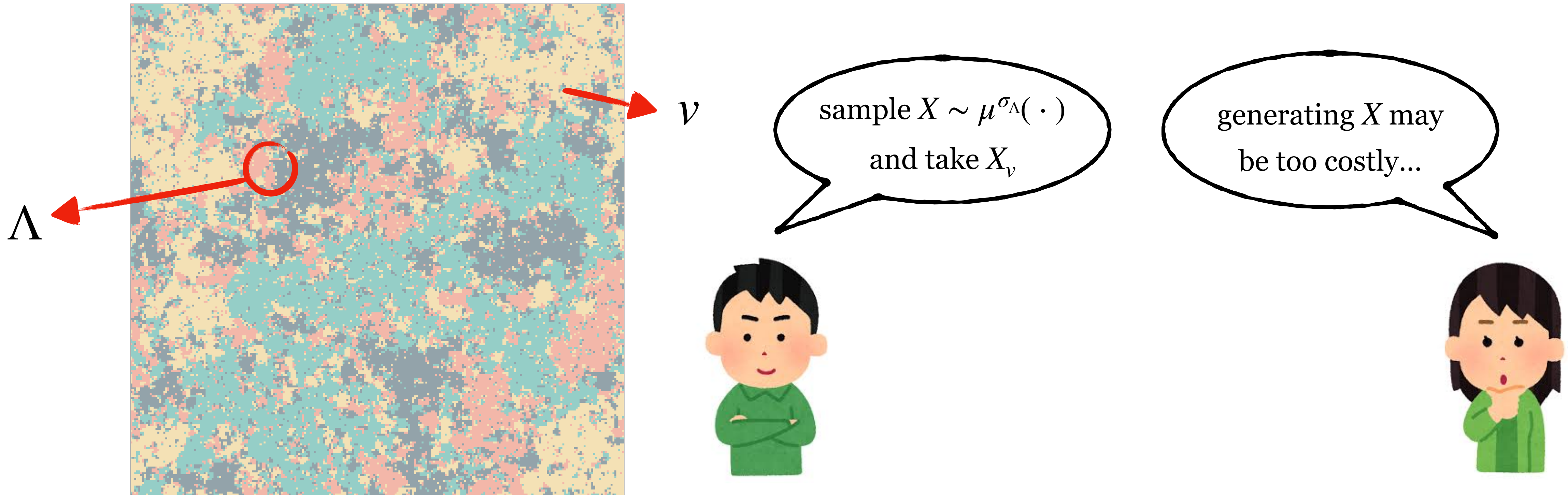
(Bayesian) probabilistic inference

Given a partial configuration σ_Λ over $\Lambda \subseteq V$ and a vertex $v \in V \setminus \Lambda$,
estimate the **conditional marginal probability distribution** $\mu_v^{\sigma_\Lambda}(\cdot)$



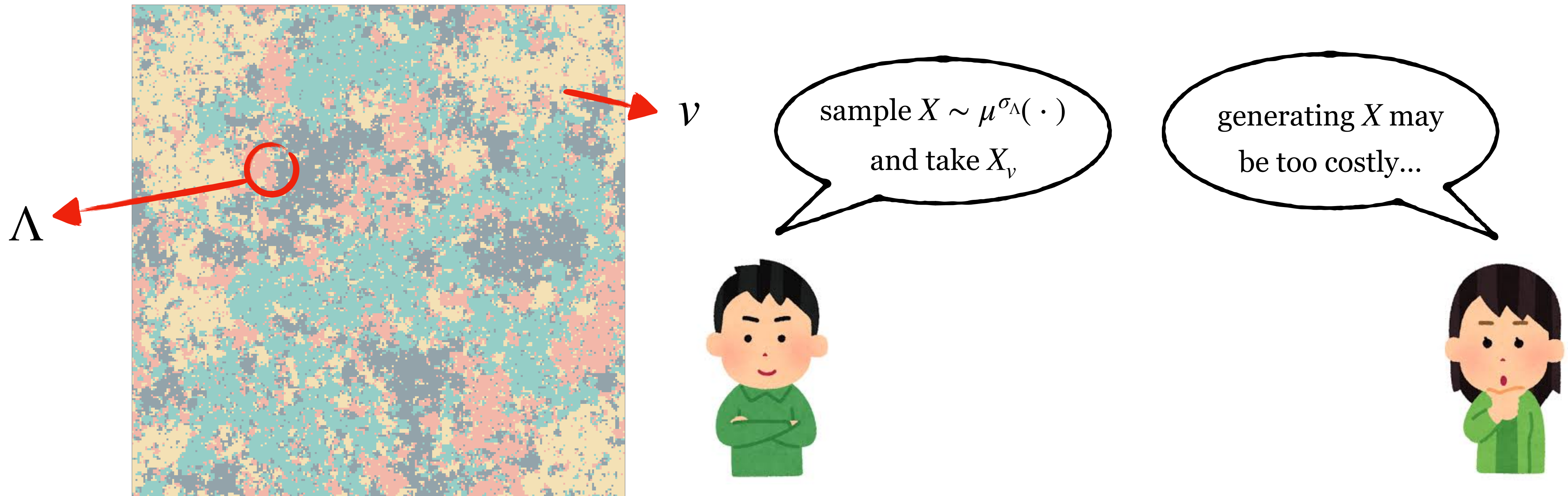
(Bayesian) probabilistic inference

Given a partial configuration σ_Λ over $\Lambda \subseteq V$ and a vertex $v \in V \setminus \Lambda$,
estimate the **conditional marginal probability distribution** $\mu_v^{\sigma_\Lambda}(\cdot)$



(Bayesian) probabilistic inference

Given a partial configuration σ_Λ over $\Lambda \subseteq V$ and a vertex $v \in V \setminus \Lambda$,
estimate the **conditional marginal probability distribution** $\mu_v^{\sigma_\Lambda}(\cdot)$



Can these **locally defined** sampling problems
be solved at at a **local** cost?

Local Sampling

Local Sampling Problem

Input: a spin system $\mathcal{S} = (G = (V, E), (\lambda_v)_{v \in V}, (A_e)_{e \in E})$, a vertex subset $\Lambda \subseteq V$

Output: a random configuration $X_\Lambda \sim \mu_\Lambda$ with local computation costs

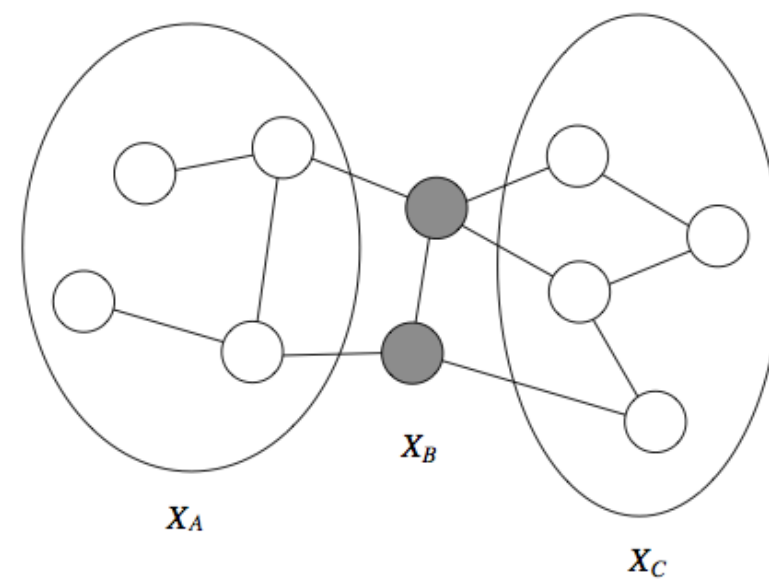
Local Sampling

Local Sampling Problem

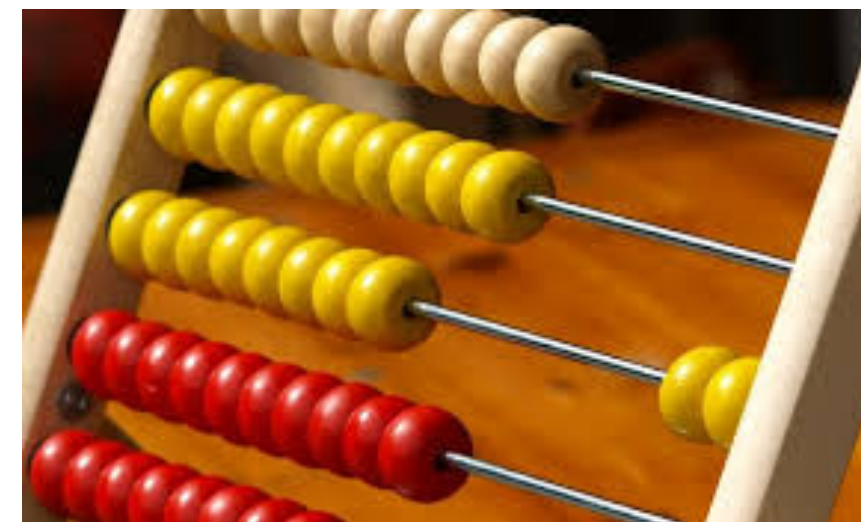
Input: a spin system $\mathcal{S} = (G = (V, E), (\lambda_v)_{v \in V}, (A_e)_{e \in E})$, a vertex subset $\Lambda \subseteq V$

Output: a random configuration $X_\Lambda \sim \mu_\Lambda$ with local computation costs

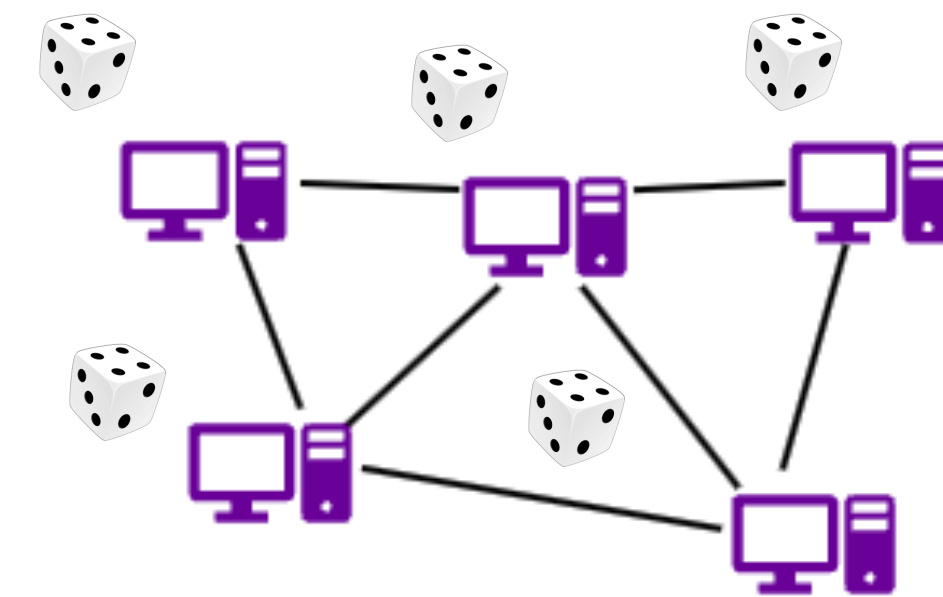
Applications:



probabilistic inference



deterministic (approx.)
counting



distributed sampling

Related definition: local access to random objects [BRY '20]
(sampling variant of Local Computation Algorithms)

Local Sampling

Local Sampling Problem

Input: a spin system $\mathcal{S} = (G = (V, E), (\lambda_v)_{v \in V}, (A_e)_{e \in E})$, a vertex subset $\Lambda \subseteq V$

Output: a random configuration $X_\Lambda \sim \mu_\Lambda$ with local computation costs

Previous local sampling algorithms [AJ '22, FGWWY '23] require
“local uniformity” (unconditional marginal lower bounds)

Local Sampling

Local Sampling Problem

Input: a spin system $\mathcal{S} = (G = (V, E), (\lambda_v)_{v \in V}, (A_e)_{e \in E})$, a vertex subset $\Lambda \subseteq V$

Output: a random configuration $X_\Lambda \sim \mu_\Lambda$ with local computation costs

Previous local sampling algorithms [AJ '22, FGWWY '23] require
“local uniformity” (unconditional marginal lower bounds)

Instance	Tractable regimes (Δ : maximum degree)	
	Global sampling	Local sampling
Ising model	¹ $\beta \in \left(\frac{\Delta - 2}{\Delta}, \frac{\Delta}{\Delta - 2} \right)$	² $\beta \in \left(1 - \frac{1}{\Theta(\Delta^2)}, 1 + \frac{1}{\Theta(\Delta^2)} \right)$
q -colorings	³ $q \geq 1.809\Delta$	N/A

¹ [Chen, Liu, Vigoda, **FOCS '20**] ² [Anand, Jerrum, **SICOMP '22**] [Feng, Guo, **W.**, Wang, Yin, **FOCS '23**] ³ [Carlson, Vigoda, **SODA '25**]

Local Sampling

Local Sampling Problem

Input: a spin system $\mathcal{S} = (G = (V, E), (\lambda_v)_{v \in V}, (A_e)_{e \in E})$, a vertex subset $\Lambda \subseteq V$

Output: a random configuration $X_\Lambda \sim \mu_\Lambda$ with local computation costs

Previous local sampling algorithms [AJ '22, FGWWY '23] require
“local uniformity” (unconditional marginal lower bounds)

Instance	Tractable regimes (Δ : maximum degree)	
	Global sampling	Local sampling
Ising model	¹ $\beta \in \left(\frac{\Delta - 2}{\Delta}, \frac{\Delta}{\Delta - 2} \right)$	² $\beta \in \left(1 - \frac{1}{\Theta(\Delta^2)}, 1 + \frac{1}{\Theta(\Delta^2)} \right)$
q -colorings	³ $q \geq 1.809\Delta$	N/A

¹ [Chen, Liu, Vigoda, **FOCS '20**] ² [Anand, Jerrum, **SICOMP '22**] [Feng, Guo, **W.**, Wang, Yin, **FOCS '23**] ³ [Carlson, Vigoda, **SODA '25**]

Is local sampling significantly harder than global sampling?

Our results

Instance	Tractable regimes		Our results (local sampling)
	Global sampling	Local sampling	
Ising model	$\beta \in \left(\frac{\Delta - 2}{\Delta}, \frac{\Delta}{\Delta - 2}\right)$	$\beta \in \left(1 - \frac{1}{\Theta(\Delta^2)}, 1 + \frac{1}{\Theta(\Delta^2)}\right)$	$\beta \in \left(\frac{\Delta - 0.5}{\Delta}, \frac{\Delta}{\Delta - 0.5}\right)$
q -colorings	$q \geq 1.809\Delta$	N/A	$q \geq 65\Delta$

Our results

Instance	Tractable regimes		Our results (local sampling)
	Global sampling	Local sampling	
Ising model	$\beta \in \left(\frac{\Delta - 2}{\Delta}, \frac{\Delta}{\Delta - 2} \right)$	$\beta \in \left(1 - \frac{1}{\Theta(\Delta^2)}, 1 + \frac{1}{\Theta(\Delta^2)} \right)$	$\beta \in \left(\frac{\Delta - 0.5}{\Delta}, \frac{\Delta}{\Delta - 0.5} \right)$
q -colorings	$q \geq 1.809\Delta$	N/A	$q \geq 65\Delta$

- **first** local sampler for **near-critical** Ising model;
- **first** local sampler for q -coloring (also **near-critical**);
- **perfect** samplers (no bias);
- expected **linear** running time: $O(\Delta \cdot |\Lambda|)$ for Ising;
 $O(\Delta^2 q \cdot |\Lambda|)$ for q -colorings.

Our results

Instance	Tractable regimes		Our results (local sampling)
	Global sampling	Local sampling	
Ising model	$\beta \in \left(\frac{\Delta - 2}{\Delta}, \frac{\Delta}{\Delta - 2} \right)$	$\beta \in \left(1 - \frac{1}{\Theta(\Delta^2)}, 1 + \frac{1}{\Theta(\Delta^2)} \right)$	$\beta \in \left(\frac{\Delta - 0.5}{\Delta}, \frac{\Delta}{\Delta - 0.5} \right)$
q -colorings	$q \geq 1.809\Delta$	N/A	$q \geq 65\Delta$

- **first** local sampler for **near-critical** Ising model;
- **first** local sampler for q -coloring (also **near-critical**);
- **perfect** samplers (no bias);
- expected **linear** running time: $O(\Delta \cdot |\Lambda|)$ for Ising;
 $O(\Delta^2 q \cdot |\Lambda|)$ for q -colorings.
- extension of the “Coupling Towards The Past” framework in [\[FGWWY’23\]](#)

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

resample $\sigma_v \sim \mu_v \left(\cdot \mid \sigma_{V \setminus \{v\}} \right)$;

return σ ;

converges to μ as $T \rightarrow \infty$!

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

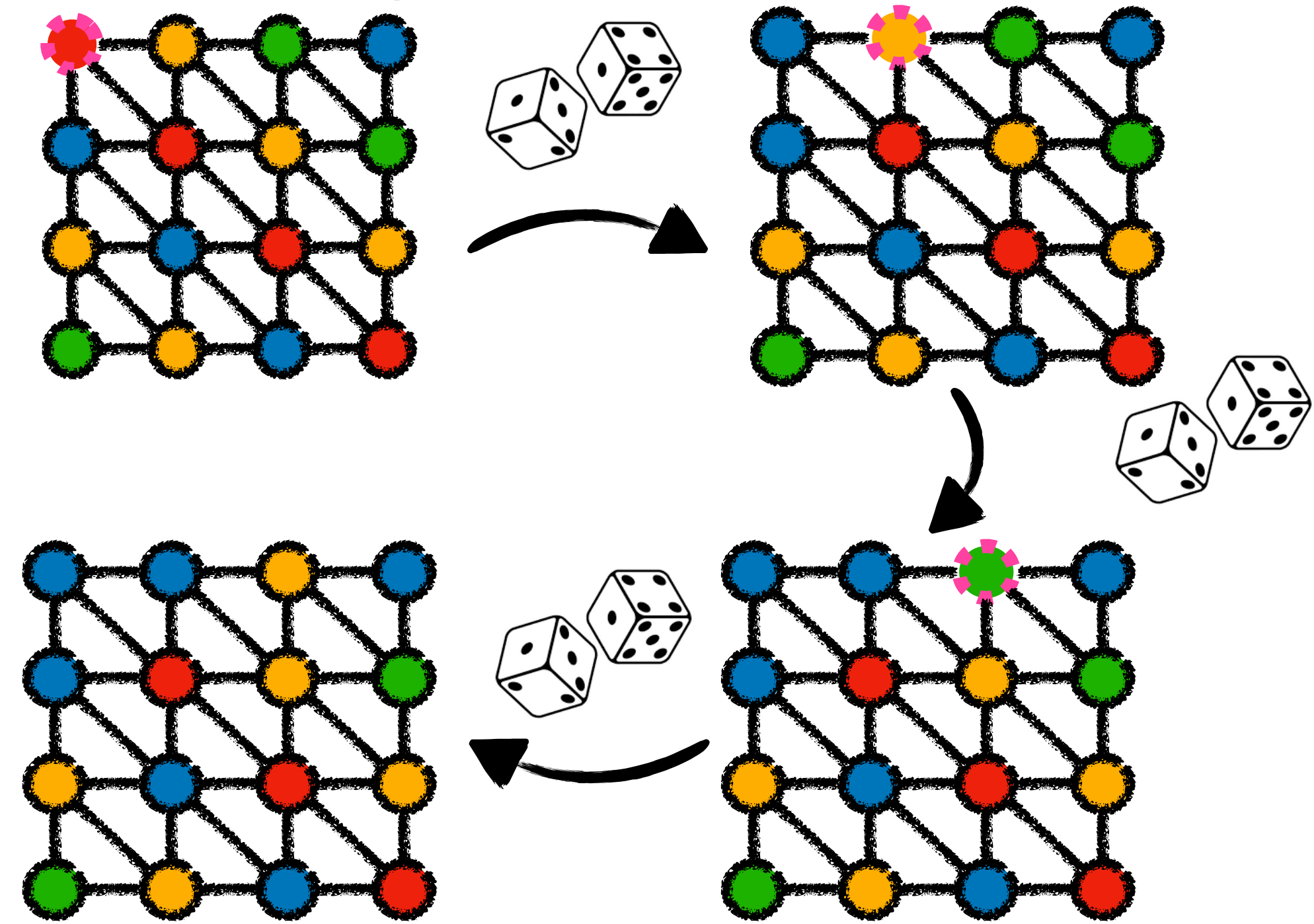
at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

resample $\sigma_v \sim \mu_v \left(\cdot \mid \sigma_{V \setminus \{v\}} \right)$;

return σ ;

converges to μ as $T \rightarrow \infty$!



Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

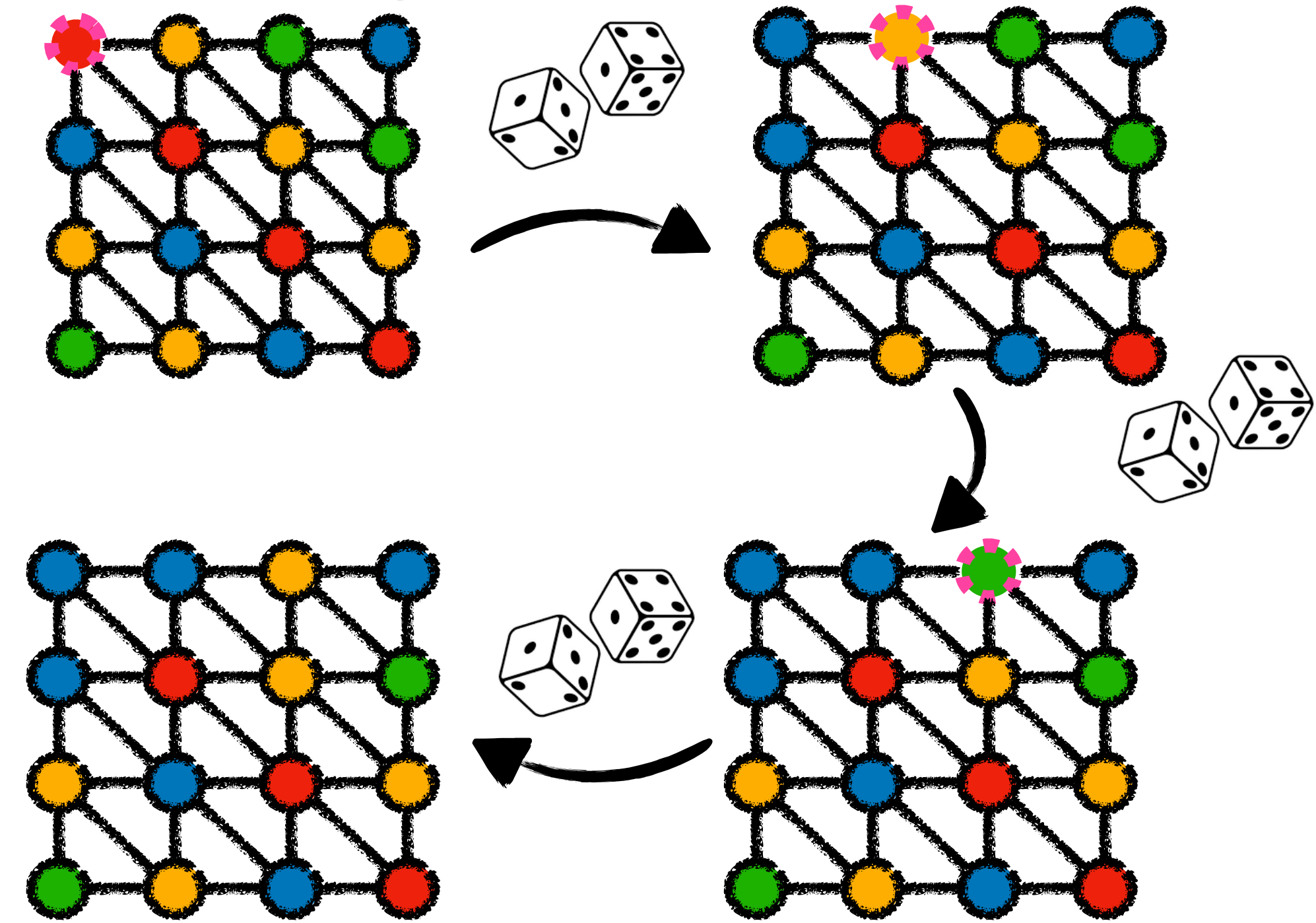
at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

resample $\sigma_v \sim \mu_v \left(\cdot \mid \sigma_{V \setminus \{v\}} \right)$;

return σ ;

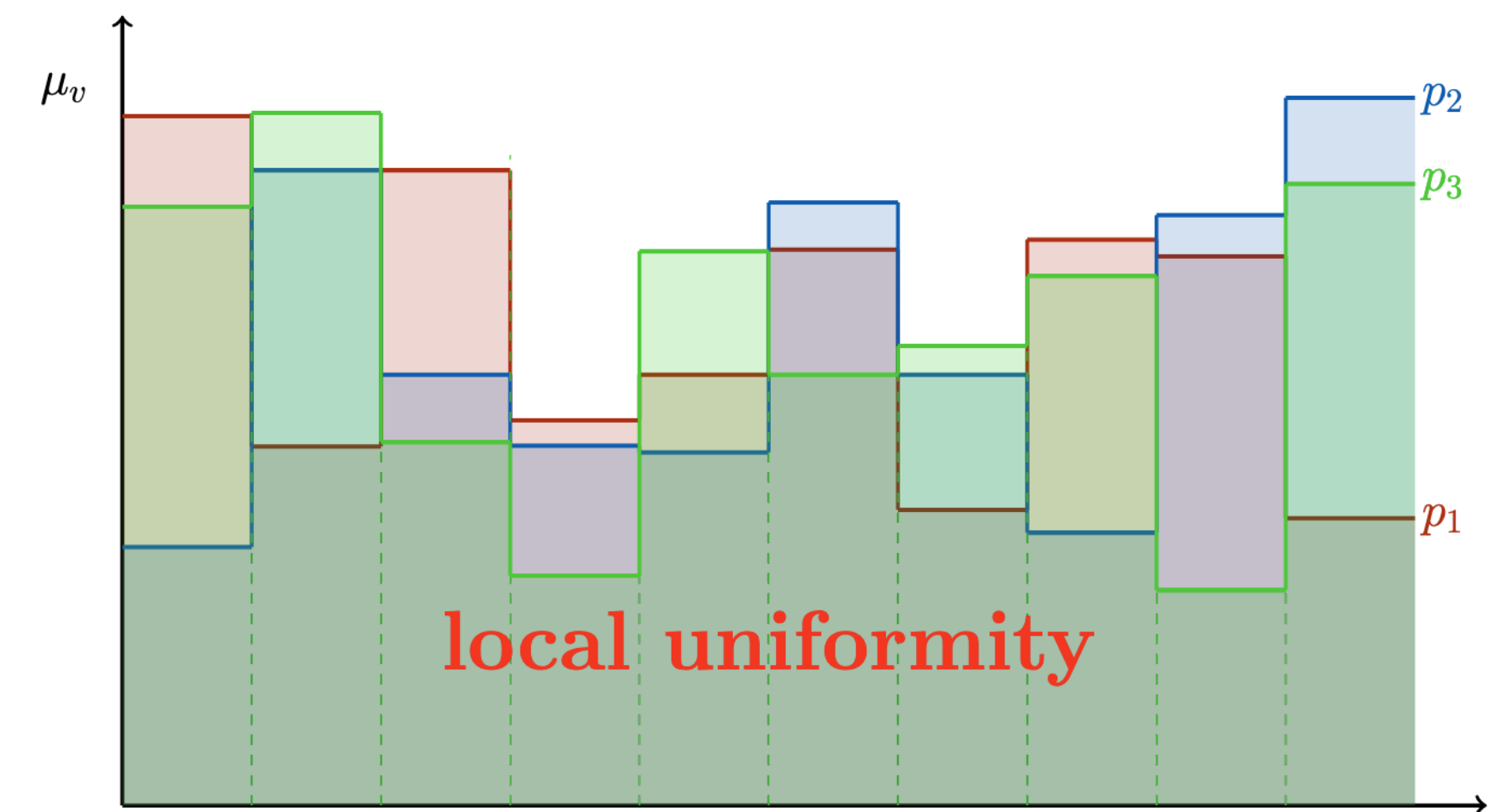
converges to μ as $T \rightarrow \infty$!



Local uniformity (marginal lower bound)

$$\forall c \in [q], \quad \mu_v^{\min}(c) \triangleq \min_{\mu(\sigma_{V \setminus \{v\}}) > 0} \mu_v \left(c \mid \sigma_{V \setminus \{v\}} \right)$$

$$\theta\text{--(locally)uniform: } \sum_{c \in [q]} \mu_v^{\min}(c) \geq \theta$$



Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

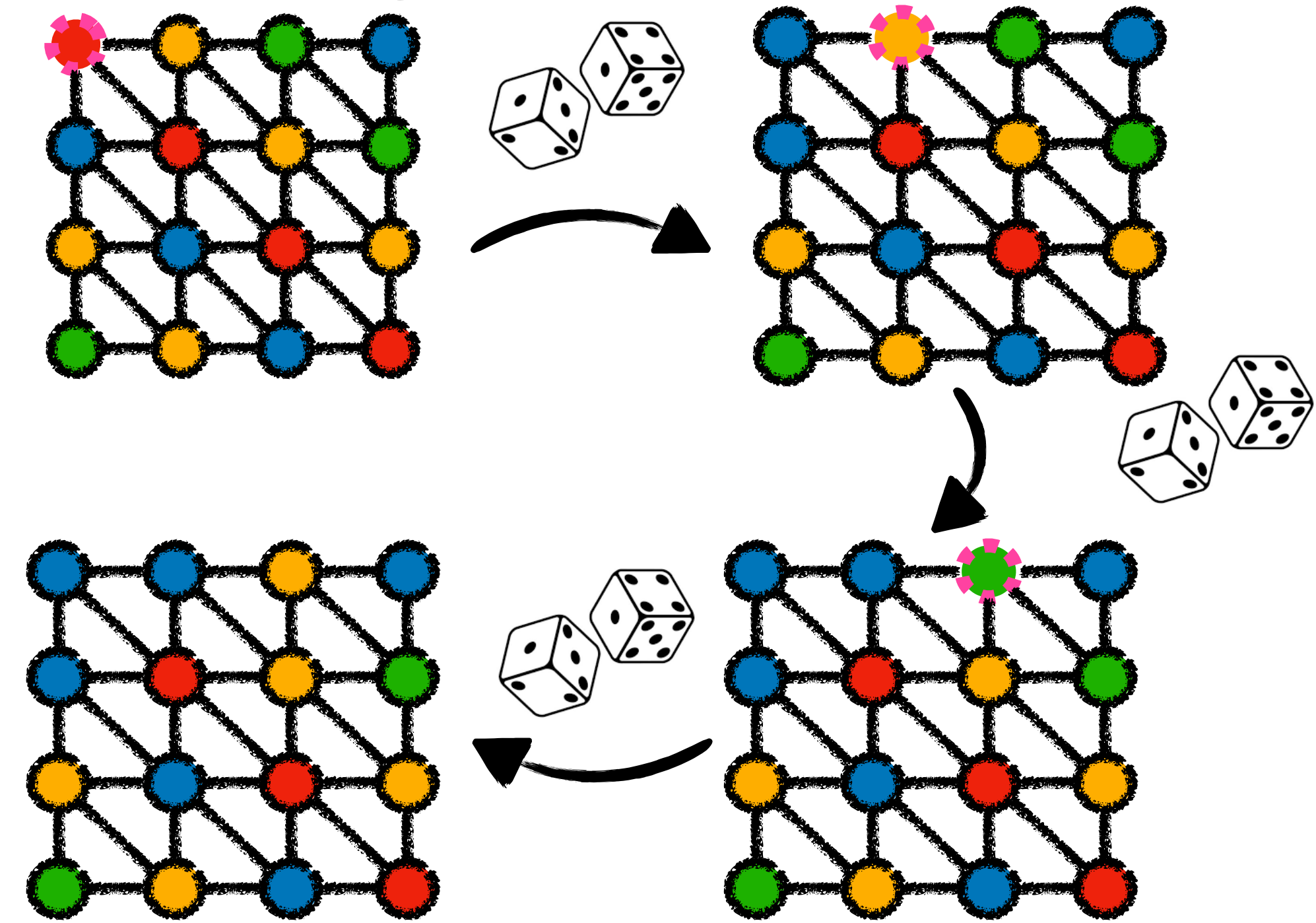
at each time $1 \leq t \leq T$:

pick the vertex $v = v_t \bmod n$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

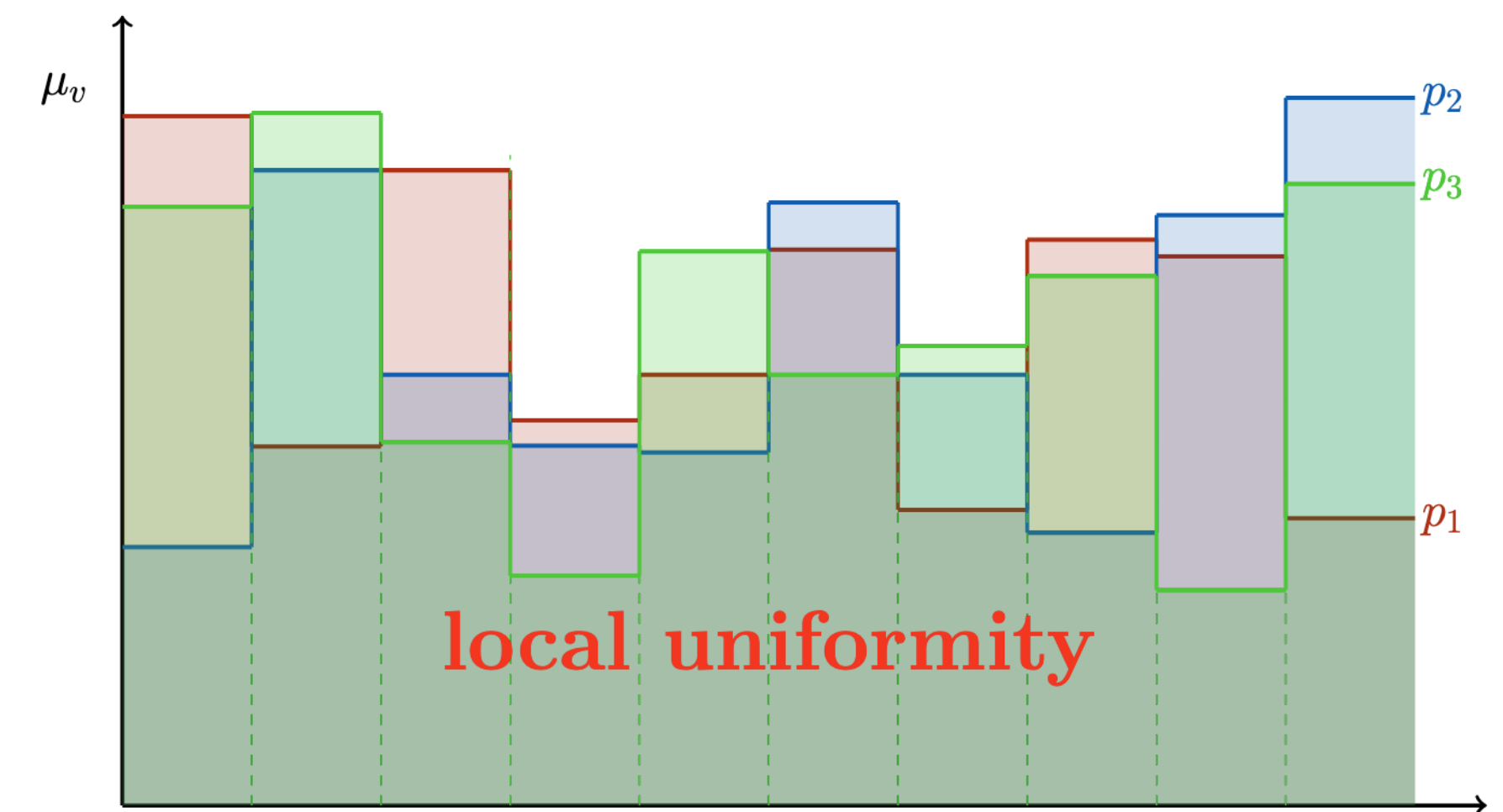
return σ ;



Local uniformity (marginal lower bound)

$$\forall c \in [q], \quad \mu_v^{\min}(c) \triangleq \min_{\mu(\sigma_{V \setminus \{v\}}) > 0} \mu_v(c \mid \sigma_{V \setminus \{v\}})$$

$$\theta\text{--(locally)uniform: } \sum_{c \in [q]} \mu_v^{\min}(c) \geq \theta$$



Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

at each time $1 \leq t \leq T$:

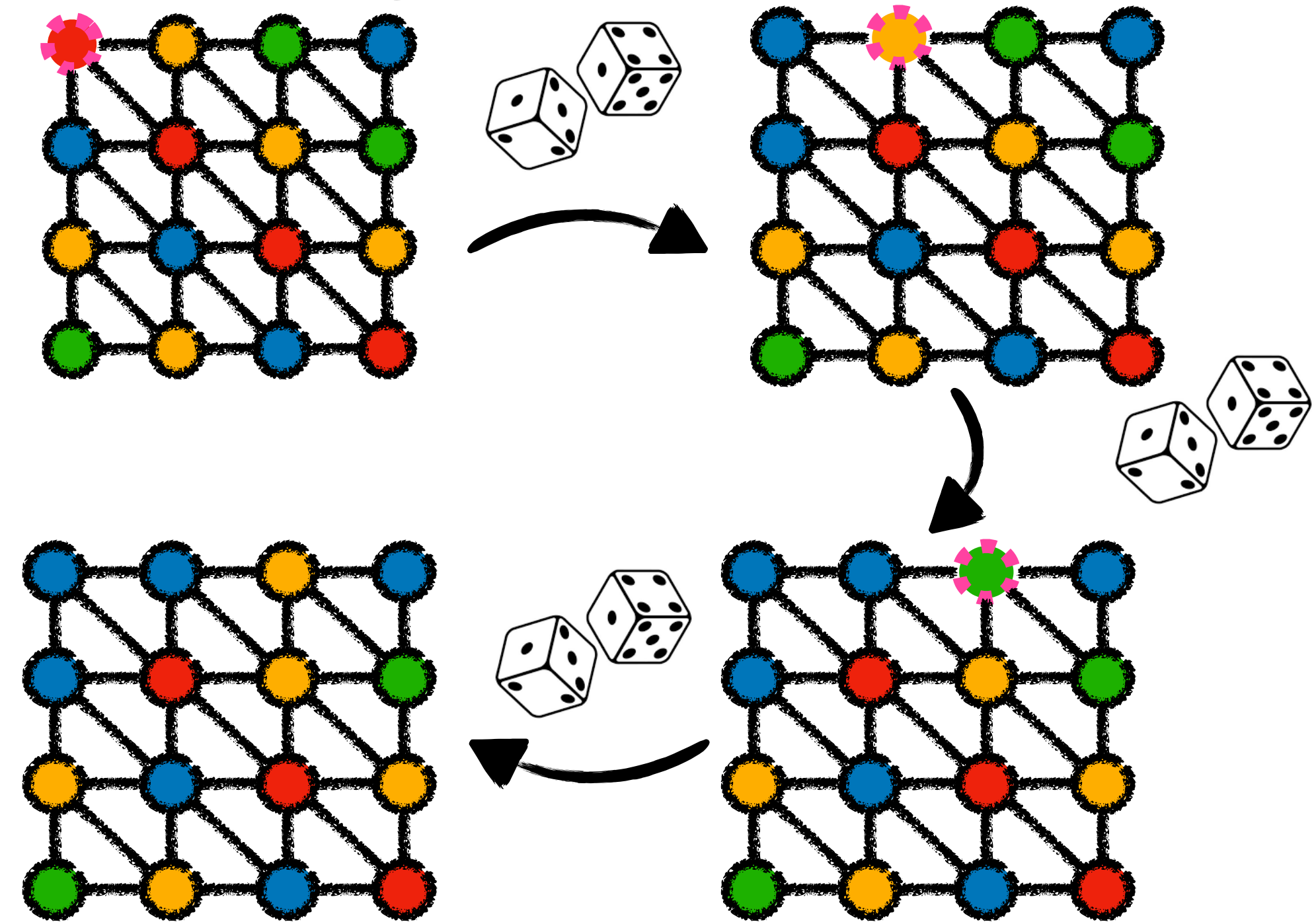
pick the vertex $v = v_t \bmod n$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ ;

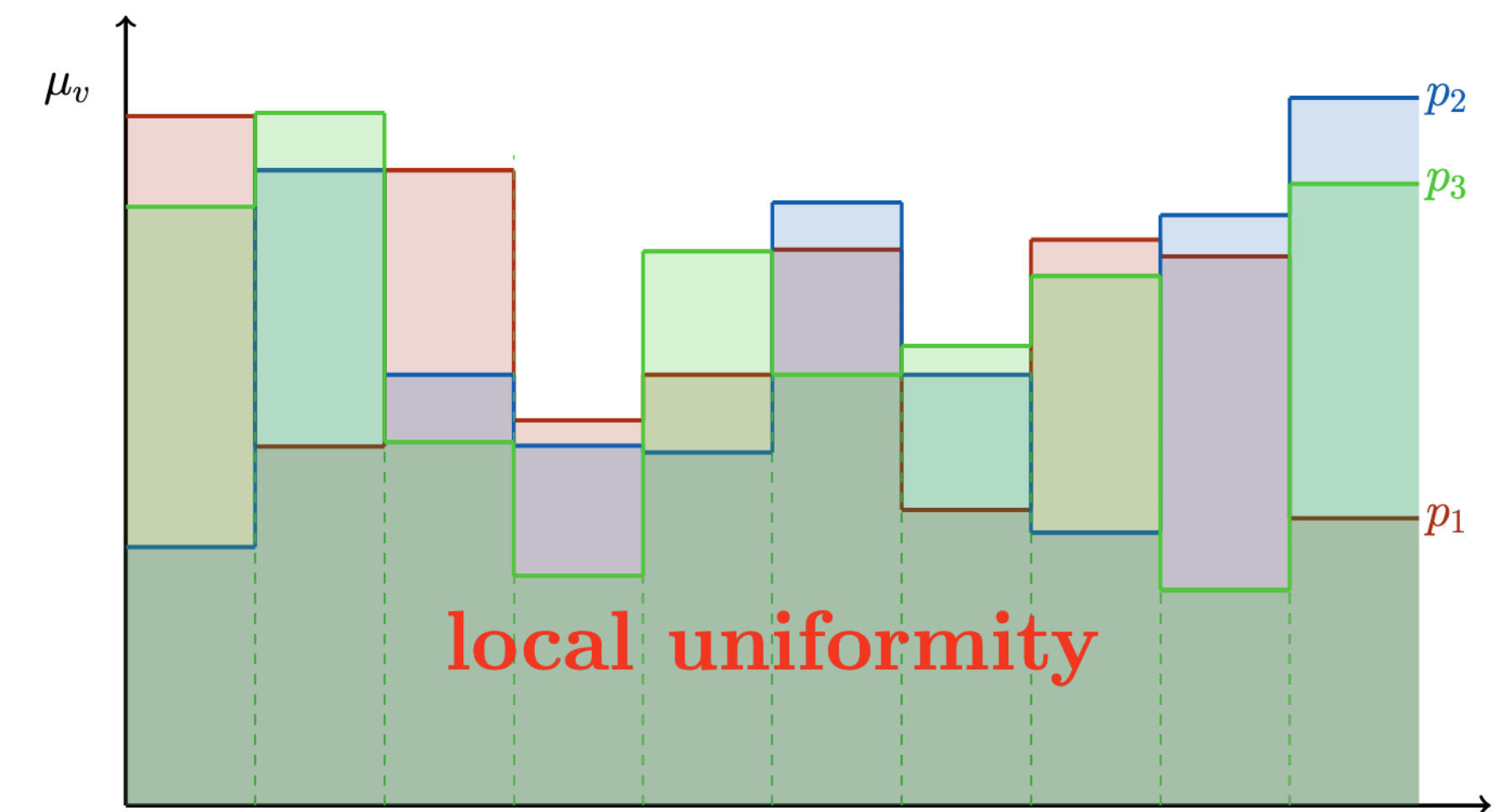
With prob. θ , update can be directly resolved!



Local uniformity (marginal lower bound)

$$\forall c \in [q], \quad \mu_v^{\min}(c) \triangleq \min_{\mu(\sigma_{V \setminus \{v\}}) > 0} \mu_v(c \mid \sigma_{V \setminus \{v\}})$$

$$\theta\text{--(locally)uniform: } \sum_{c \in [q]} \mu_v^{\min}(c) \geq \theta$$



Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

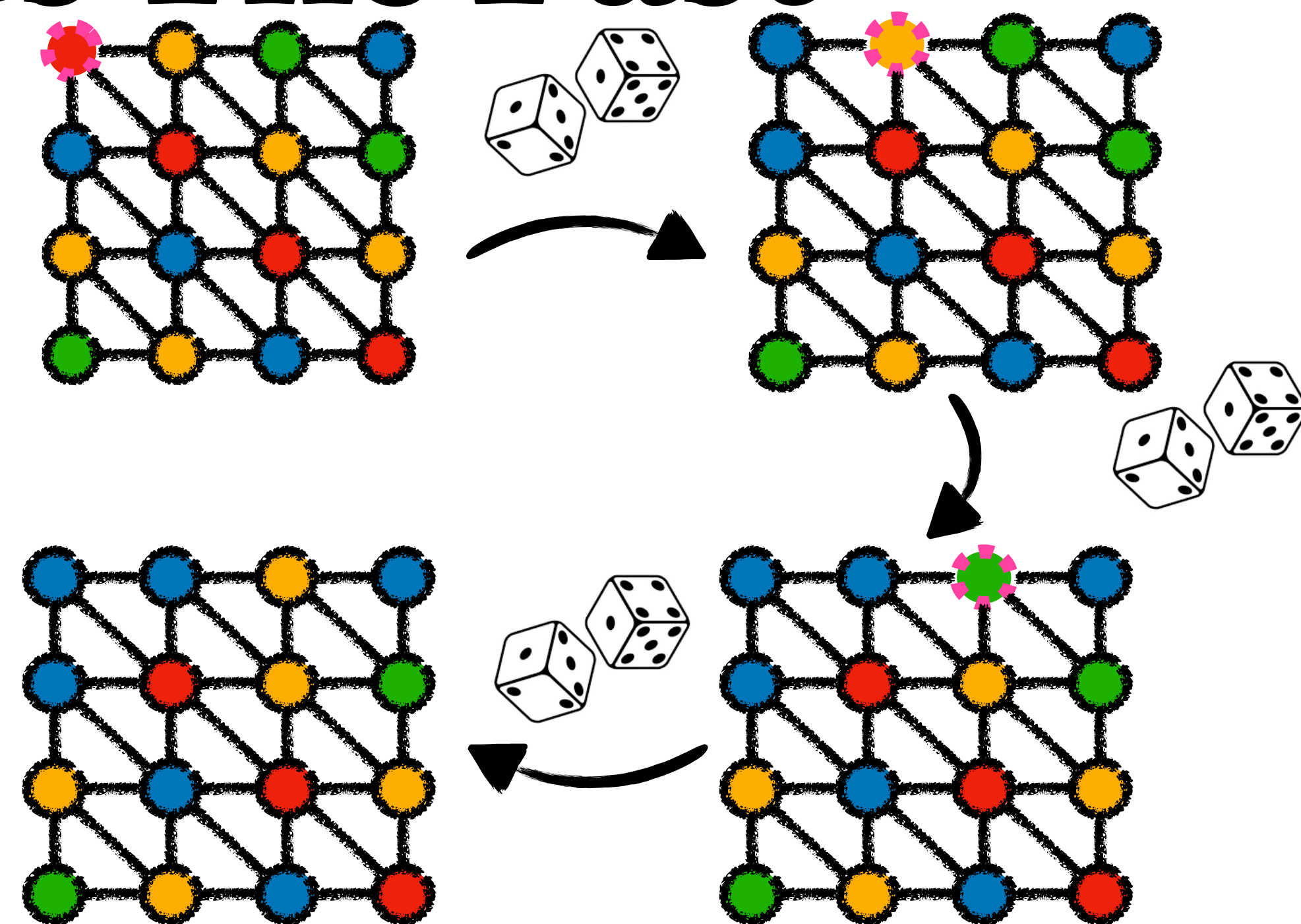
at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ ;



t	...	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	...	?	?	?	?	?	?	?	?	?
outcome	...	?	?	?	?	?	?	?	?	?

final state $\rightarrow$ (approx) local sampler

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

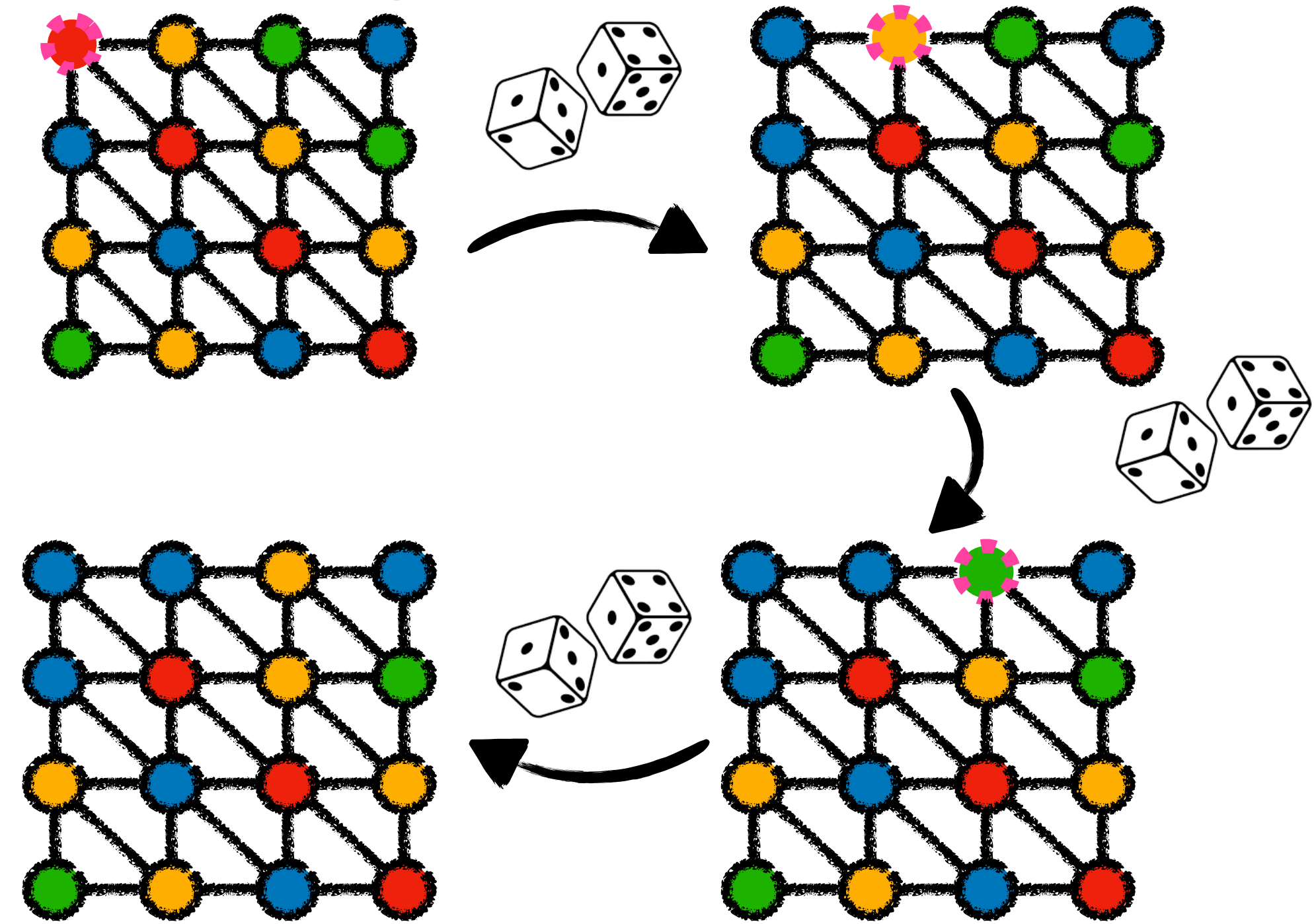
at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ ;



t	...	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	...	?	?	?	?	?	?	?	?	?
outcome	...	?	?	?	?	?	?	?	?	?

final state $\rightarrow$ (approx) local sampler



backward deduction!

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

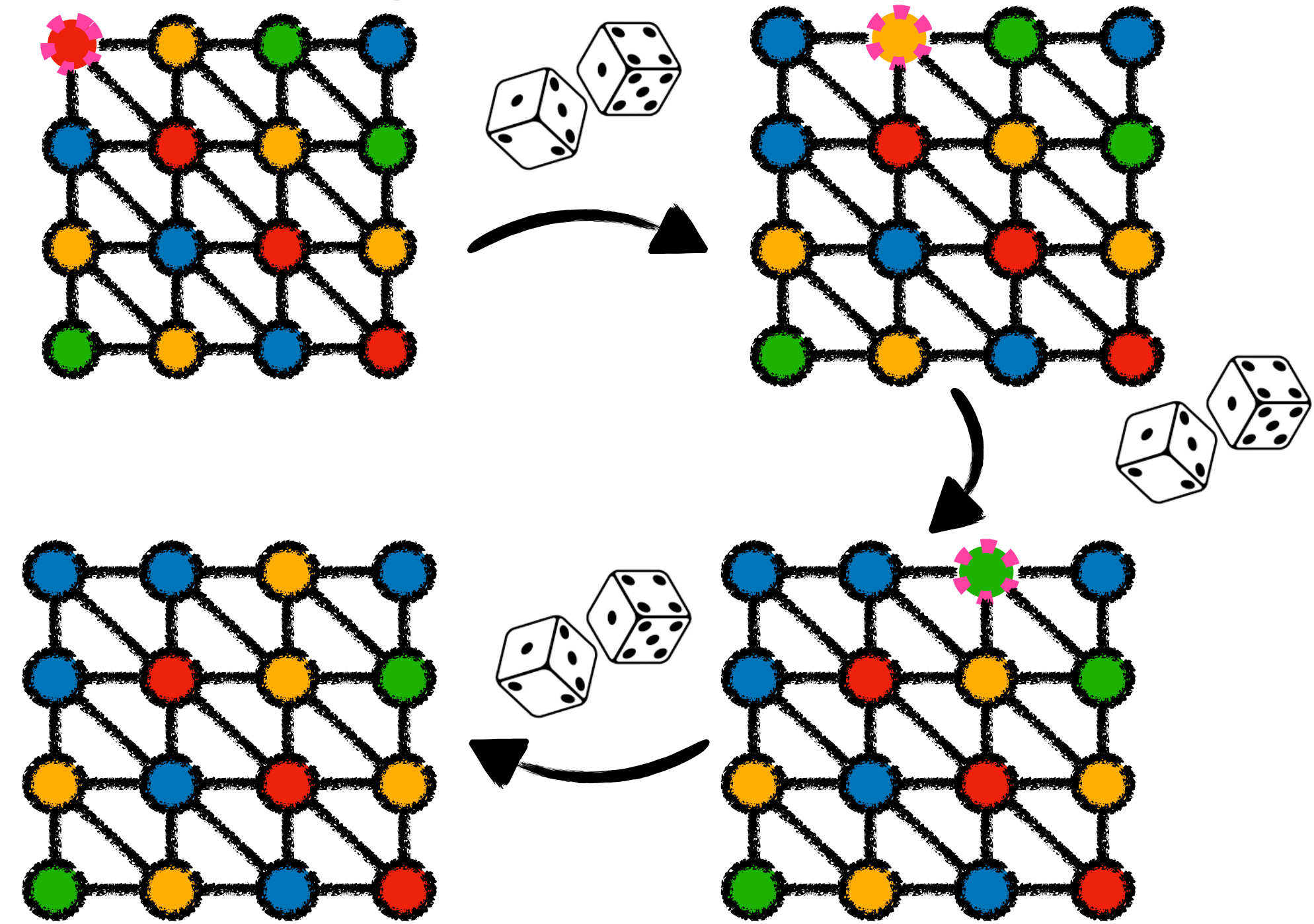
at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ ;



t	...	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	...	?	?	?	?	?	?	?	?	?
outcome	...	?	?	?	?	?	?	?	?	?

final state $\rightarrow$ (approx) local sampler



backward deduction!

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

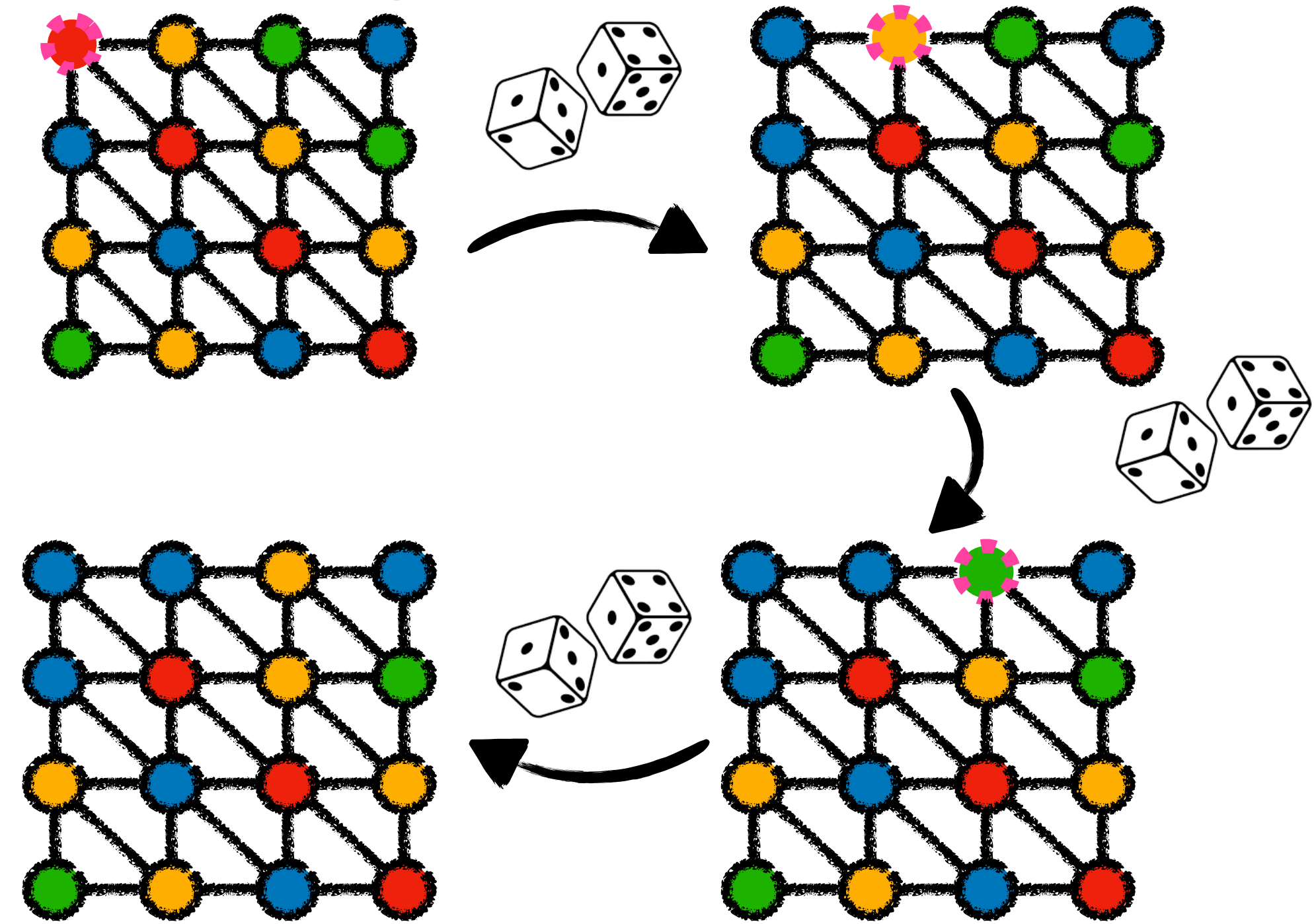
at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ ;



t	...	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	...	?	?	?	?	?	?	?	?	1
outcome	...	?	?	?	?	?	?	?	?	

final state $\rightarrow$ (approx) local sampler



backward deduction!

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

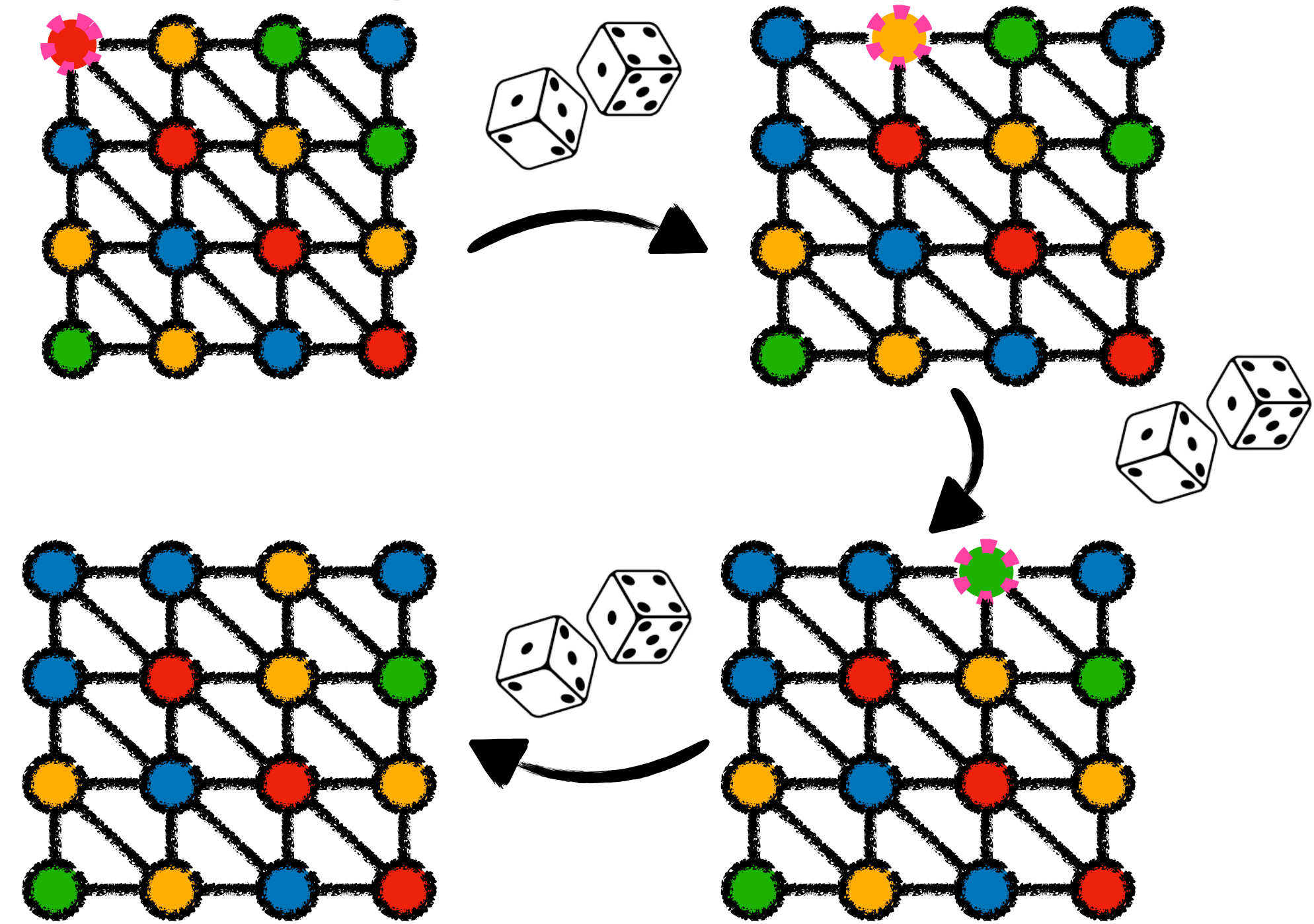
at each time $1 \leq t \leq T$:

pick the vertex $v = v_t \bmod n$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ ;



t	...	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	...	?	?	?	?	?	?	?	?	0
outcome	...	?	?	?	?	?	?	?	?	?

final state $\rightarrow$ (approx) local sampler



backward deduction!

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

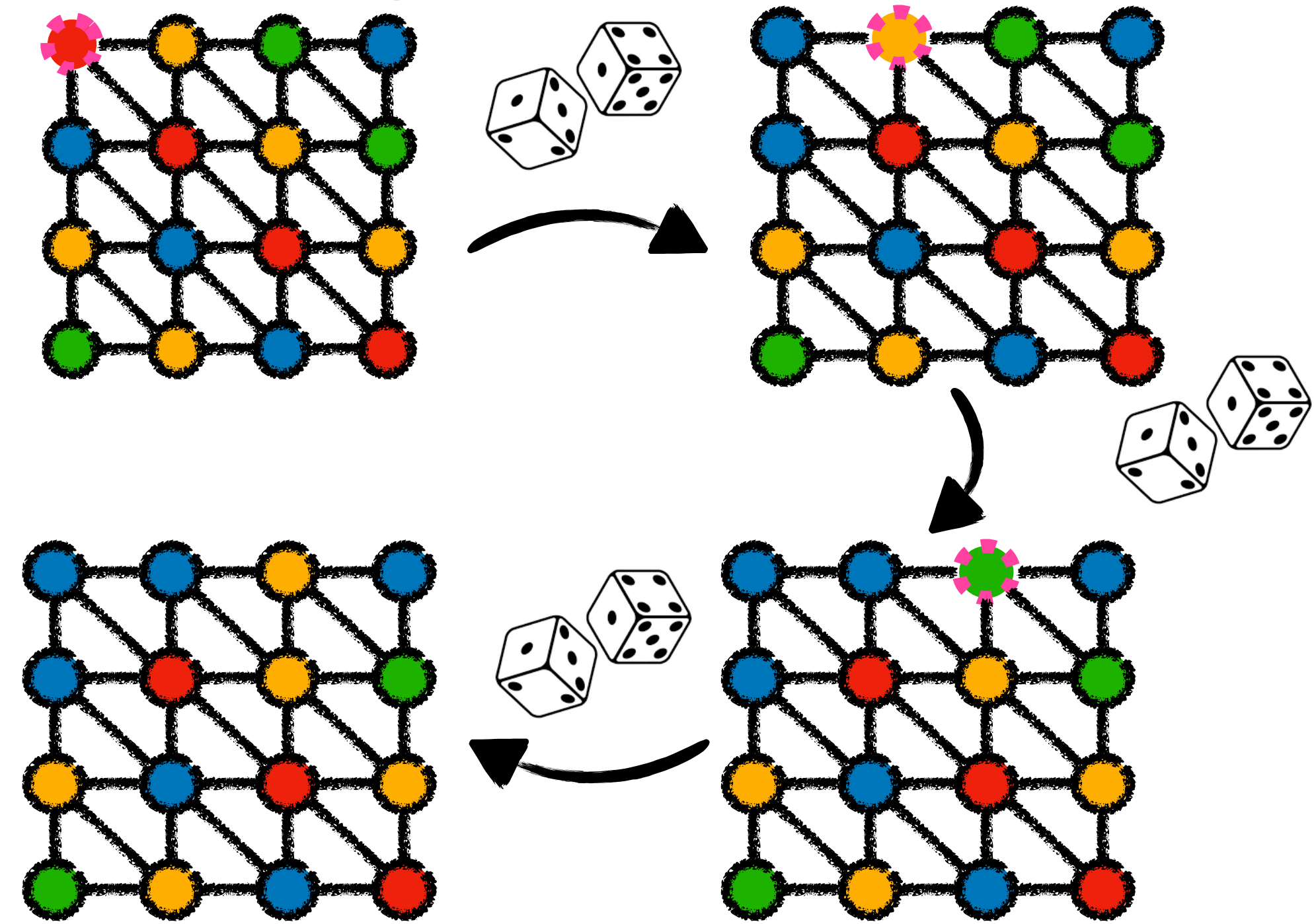
at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ ;



t	...	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	...	?	1	?	?	1	?	1	?	0
outcome	...	?		?	?		?		?	

final state $\rightarrow$ (approx) local sampler



backward deduction!

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

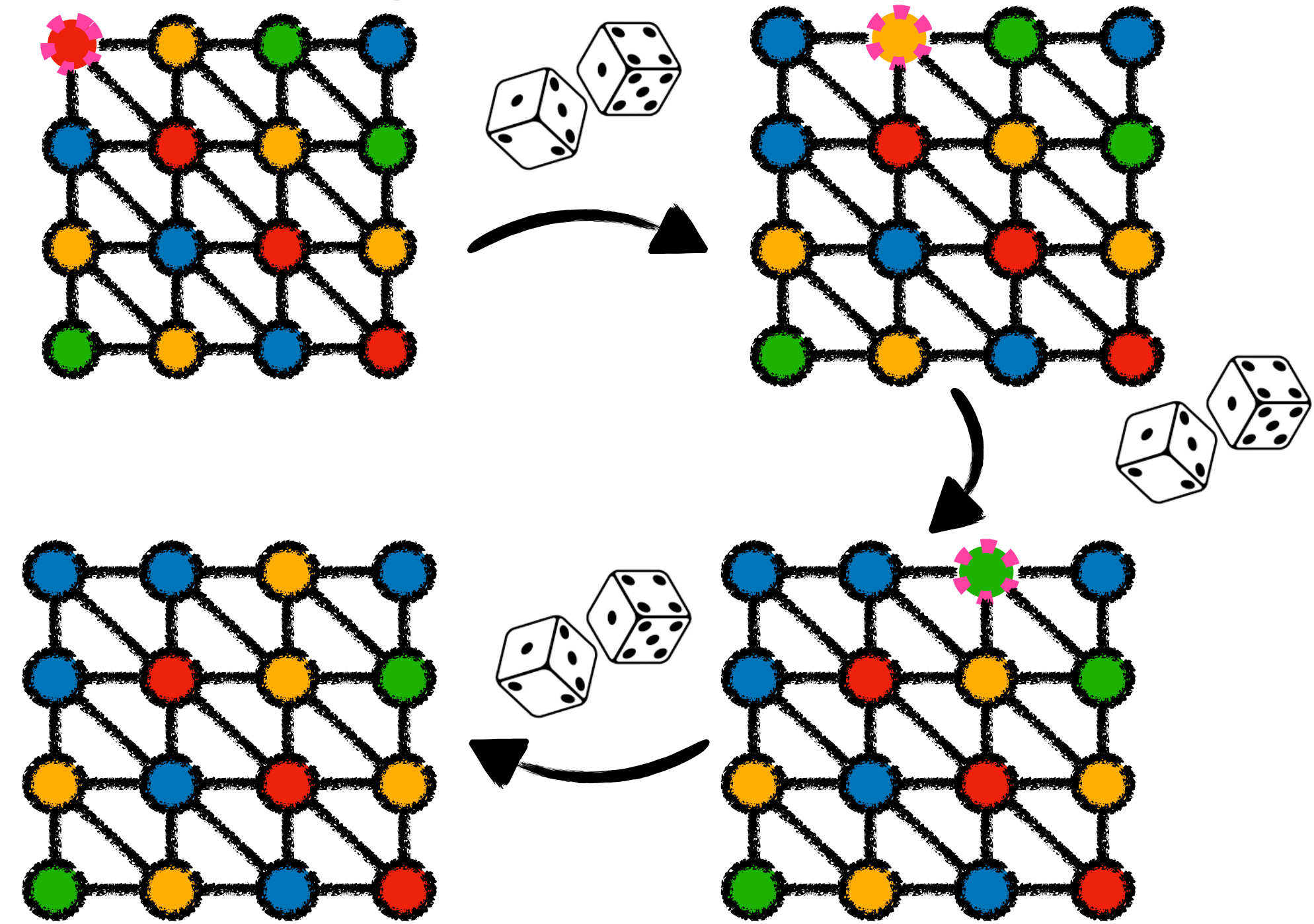
at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ ;



t	...	$T-8$	$T-7$	$T-6$	$T-5$	$T-4$	$T-3$	$T-2$	$T-1$	T
r_t	...	?	1	?	?	1	?	1	?	0
outcome	...	?		?	?		?		?	

final state $\rightarrow$ (approx) local sampler



backward deduction!



Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

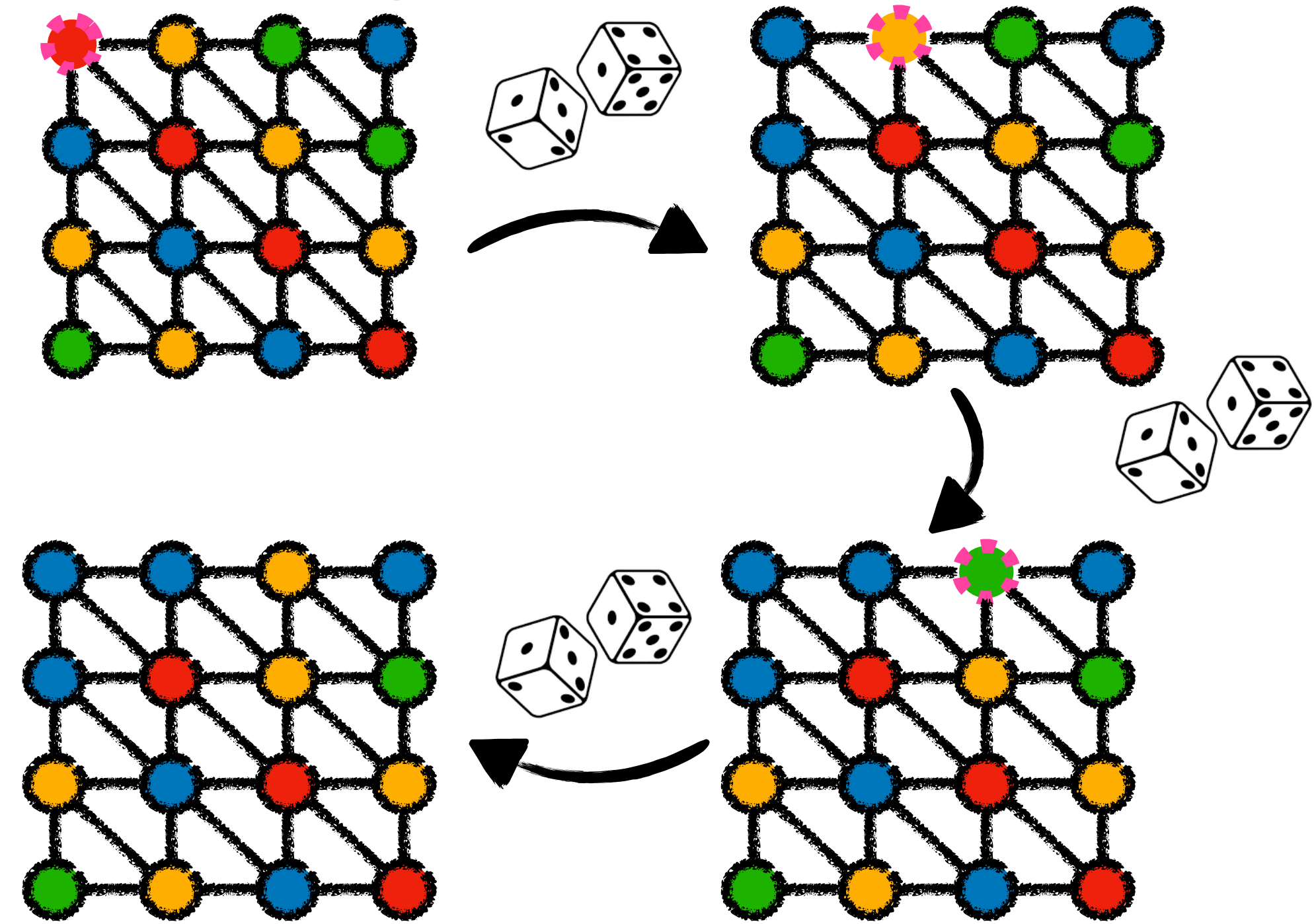
at each time $1 \leq t \leq T$:

pick the vertex $v = v_t \bmod n$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ ;



t	...	-8	-7	-6	-5	-4	-3	-2	-1	0
r_t	...	?	1	?	?	1	?	1	?	0
outcome	...	?		?	?		?		?	

final state $\rightarrow$ perfect local sampler

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

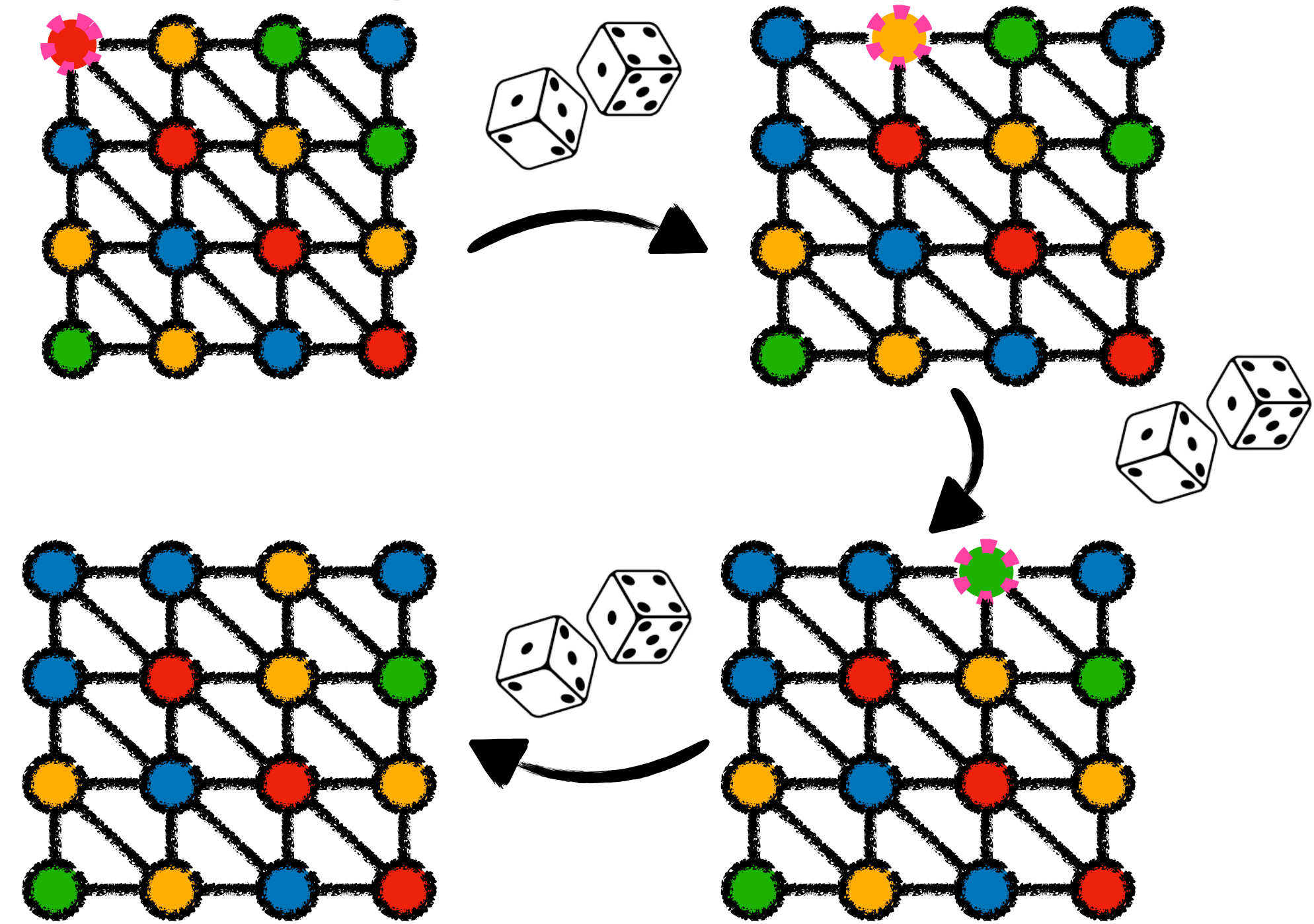
at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ ;



t	...	-8	-7	-6	-5	-4	-3	-2	-1	0
r_t	...	?	1	?	?	1	?	1	?	0
outcome	...	?		?	?		?		?	

final state $\rightarrow$ **perfect** local sampler

need $(1 - \theta)\Delta \leq 1$ to terminate

Coupling Towards The Past

(Systematic scan) Glauber dynamics

start with arbitrary configuration σ with $w(\sigma) > 0$;

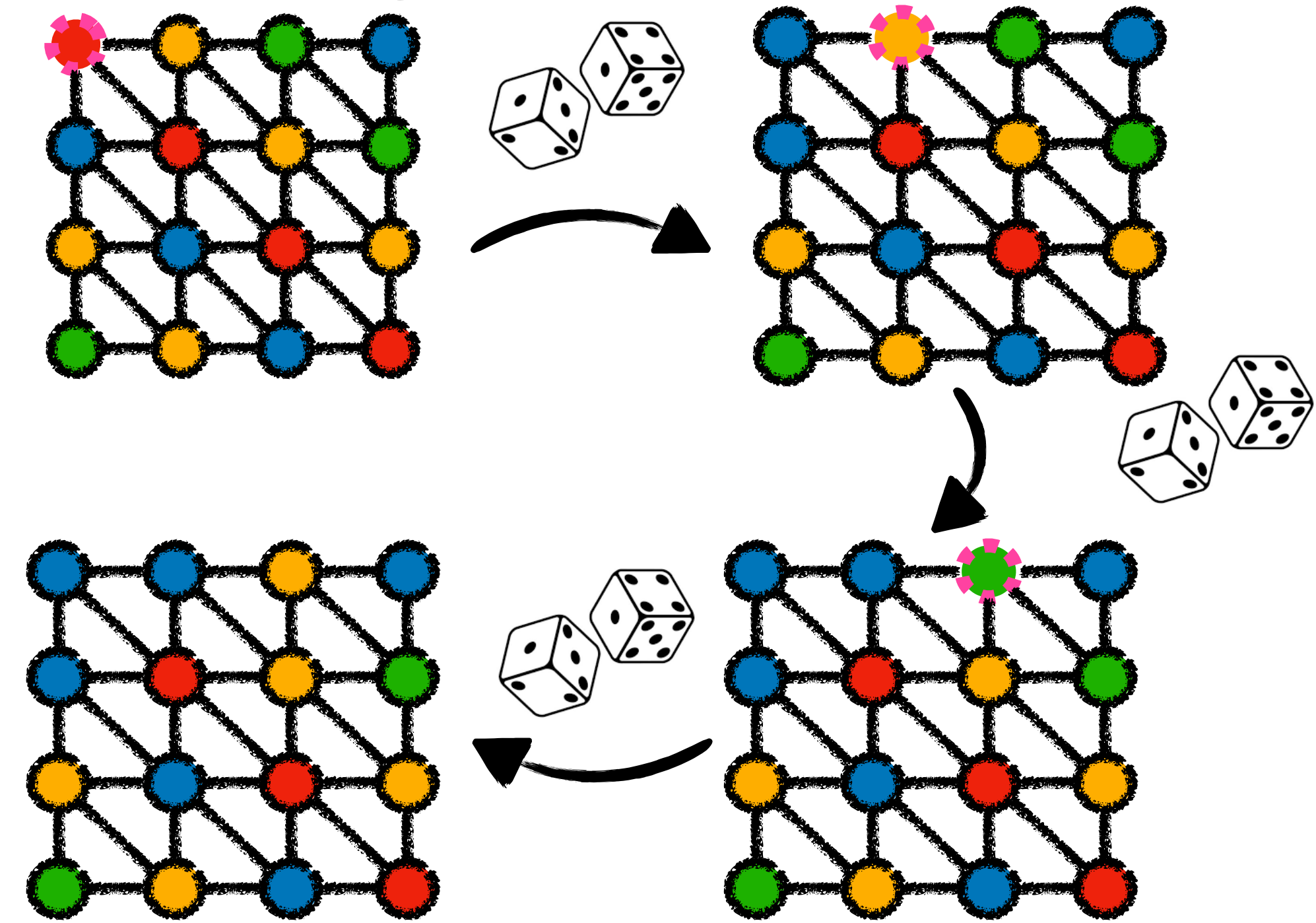
at each time $1 \leq t \leq T$:

pick the vertex $v = v_{t \bmod n}$; (assume $V = \{v_0, \dots, v_{n-1}\}$)

sample $r_t \sim \text{Ber}(\theta)$ and resample σ_v such that

$$\Pr[\sigma_v = c] \propto \begin{cases} \mu_v^{\min}(c) & r_t = 1 \\ \mu_v(c \mid \sigma_{V \setminus \{v\}}) - \mu_v^{\min}(c) & r_t = 0 \end{cases}$$

return σ ;



t	...	-8	-7	-6	-5	-4	-3	-2	-1	0
r_t	...	?	1	?	?	1	?	1	?	0
outcome	...	?		?	?		?		?	

final state $\rightarrow$ **perfect** local sampler

need $(1 - \theta)\Delta \leq 1$ to terminate

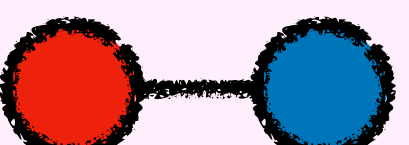
hidden (default) **grand coupling!**

Warmup: Local Ising Sampler

Ising model

two states  

external fields  λ  1

interaction   β
  1

marginal

$$\Pr[\text{red}] = \frac{\lambda \beta^{\#N(\text{red})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

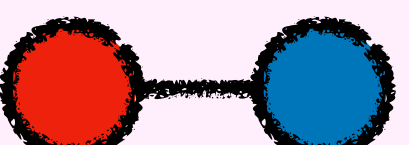
$$\Pr[\text{blue}] = \frac{\beta^{\#N(\text{blue})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

Warmup: Local Ising Sampler

Ising model

two states  

external fields  λ  1

interaction   β
  1

marginal

$$\Pr[\text{red}] = \frac{\lambda \beta^{\#N(\text{red})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

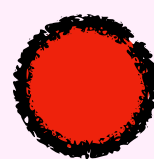
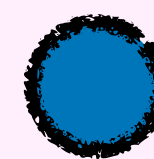
$$\Pr[\text{blue}] = \frac{\beta^{\#N(\text{blue})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

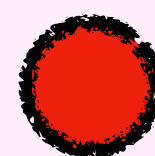
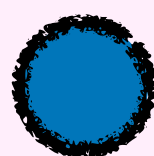


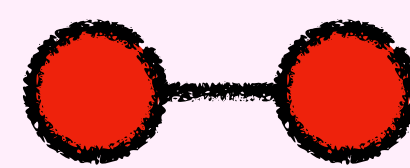
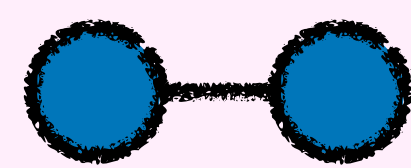
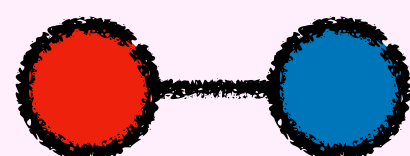
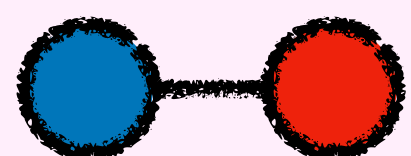
need $\mathbb{E}[\text{neighbors checked}] \leq 1$

Warmup: Local Ising Sampler

Ising model

two states  

external fields  λ  1

interaction  β 
 1 

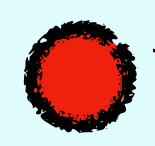
marginal

$$\Pr[\text{red}] = \frac{\lambda \beta^{\#N(\text{red})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

$$\Pr[\text{blue}] = \frac{\beta^{\#N(\text{blue})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

A rejection sampler for $\beta \leq 1$

for $i = 1, 2, \dots$:

propose c_i :  w.p. $\frac{\lambda}{1 + \lambda}$;  w.p. $\frac{1}{1 + \lambda}$;

for each $u \in N(v)$ uniformly sample $r_{i,u} \in [0, 1]$;

accept iff $r_{i,u} < \beta$ for each u with same color;

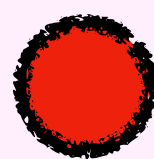
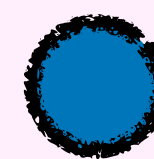
$$\Pr[\text{accept}] \propto \beta^{\#N(c_i)}!$$

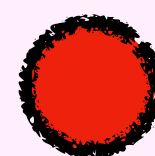
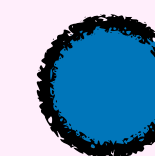


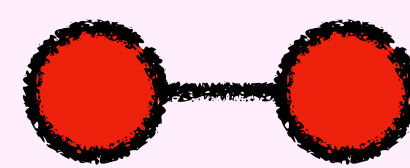
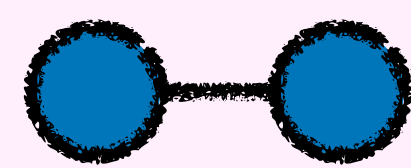
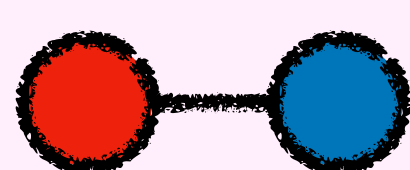
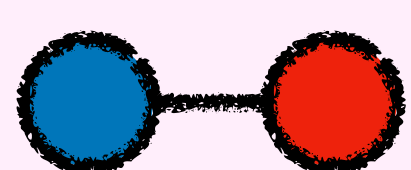
need $\mathbb{E}[\text{neighbors checked}] \leq 1$

Warmup: Local Ising Sampler

Ising model

two states  

external fields  λ  1

interaction   β
  1

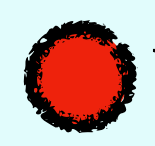
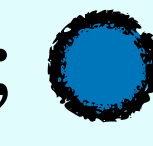
marginal

$$\Pr[\text{red}] = \frac{\lambda \beta^{\#N(\text{red})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

$$\Pr[\text{blue}] = \frac{\beta^{\#N(\text{blue})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

A rejection sampler for $\beta \leq 1$

for $i = 1, 2, \dots$:

propose c_i :  w.p. $\frac{\lambda}{1 + \lambda}$;  w.p. $\frac{1}{1 + \lambda}$;

for each $u \in N(v)$ uniformly sample $r_{i,u} \in [0, 1]$;

accept iff $r_{i,u} < \beta$ for each u with same color;

$$\Pr[\text{accept}] \propto \beta^{\#N(c_i)}!$$

Resolve an update

for $i = 1, 2, \dots$

for each $u \in N(v)$:

if $r_{i,u} \geq \beta$:

resolve u , go to next outer iter. if same color

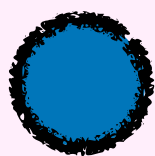
return c_i

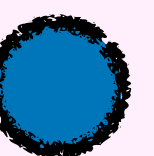


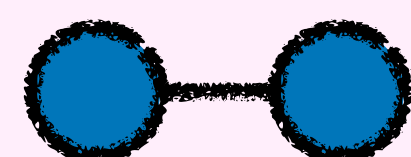
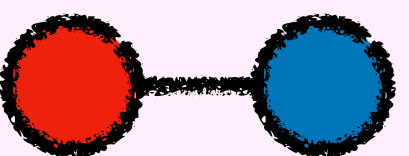
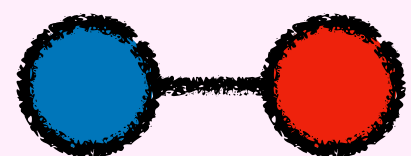
need $\mathbb{E}[\text{neighbors checked}] \leq 1$

Warmup: Local Ising Sampler

Ising model

two states  

external fields  λ  1

interaction   β
  1

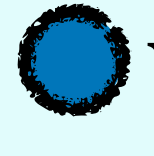
marginal

$$\Pr[\text{red}] = \frac{\lambda \beta^{\#N(\text{red})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

$$\Pr[\text{blue}] = \frac{\beta^{\#N(\text{blue})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

A rejection sampler for $\beta \geq 1$

for $i = 1, 2, \dots$:

propose c_i :  w.p. $\frac{\lambda}{1 + \lambda}$;  w.p. $\frac{1}{1 + \lambda}$;

for each $u \in N(v)$ uniformly sample $r_{i,u} \in [0, 1]$;

accept iff $r_{i,u} < 1/\beta$ for each u with opposite color;

$$\Pr[\text{accept}] \propto \beta^{\#N(c_i)}!$$

Resolve an update

for $i = 1, 2, \dots$:

for each $u \in N(v)$:

if $r_{i,u} \geq 1/\beta$:

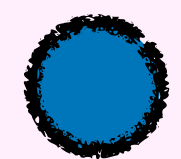
resolve u , go to next outer iter. if opposite color;
 return c_i ;

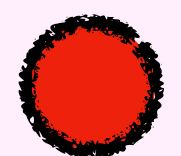
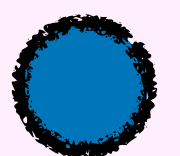


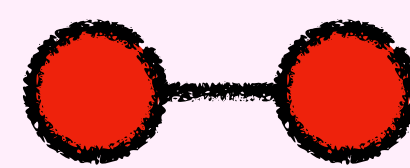
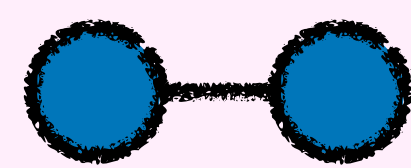
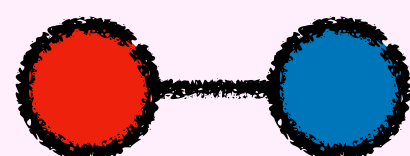
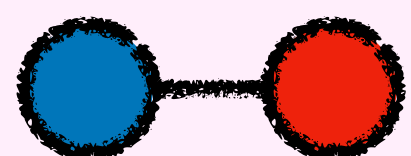
need $\mathbb{E}[\text{neighbors checked}] \leq 1$

Warmup: Local Ising Sampler

Ising model

two states  

external fields  λ  1

interaction  β 
 1 

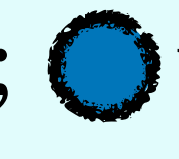
marginal

$$\Pr[\text{red}] = \frac{\lambda \beta^{\#N(\text{red})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

$$\Pr[\text{blue}] = \frac{\beta^{\#N(\text{blue})}}{\lambda \beta^{\#N(\text{red})} + \beta^{\#N(\text{blue})}}$$

A rejection sampler for $\beta \geq 1$

for $i = 1, 2, \dots$:

propose c_i :  w.p. $\frac{\lambda}{1 + \lambda}$;  w.p. $\frac{1}{1 + \lambda}$;

for each $u \in N(v)$ uniformly sample $r_{i,u} \in [0, 1]$;

accept iff $r_{i,u} < 1/\beta$ for each u with **opposite** color;

$$\Pr[\text{accept}] \propto \beta^{\#N(c_i)}!$$

Resolve an update

for $i = 1, 2, \dots$:

for each $u \in N(v)$:

if $r_{i,u} \geq 1/\beta$:

resolve u , go to next outer iter. if **opposite** color;
 return c_i ;



need $\mathbb{E}[\text{neighbors checked}] \leq 1$

smoother deduction rules for *soft* constraints!

Local q -coloring Sampler

Colorings

$$\text{marginal } \Pr[\text{red}] = \begin{cases} 0 & \text{red} \in S_t \\ \frac{1}{|[q] \setminus S_t|} & \text{red} \notin S_t \end{cases}$$

S_t = set of neighboring colors

Local q -coloring Sampler

Colorings

$$\text{marginal } \Pr[\textcolor{red}{\bullet}] = \begin{cases} 0 & \textcolor{red}{\bullet} \in S_t \\ \frac{1}{|[q] \setminus S_t|} & \textcolor{red}{\bullet} \notin S_t \end{cases}$$

S_t = set of neighboring colors

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color $\textcolor{red}{\bullet}$ from $[q]$;

accept $\textcolor{red}{\bullet}$ if $\textcolor{red}{\bullet} \notin S_t$;

Local q -coloring Sampler

Colorings

$$\text{marginal } \Pr[\textcolor{red}{\bullet}] = \begin{cases} 0 & \textcolor{red}{\bullet} \in S_t \\ \frac{1}{|[q] \setminus S_t|} & \textcolor{red}{\bullet} \notin S_t \end{cases}$$

S_t = set of neighboring colors

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color $\textcolor{red}{\bullet}$ from $[q]$;

accept $\textcolor{red}{\bullet}$ if $\textcolor{red}{\bullet} \notin S_t$;



Local q -coloring Sampler

Colorings

$$\text{marginal } \Pr[\text{red}] = \begin{cases} 0 & \text{red} \in S_t \\ \frac{1}{|[q] \setminus S_t|} & \text{red} \notin S_t \end{cases}$$

S_t = set of neighboring colors

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color  from $[q]$;

accept  if  $\notin S_t$;



need to check at
least one neighbor...

$$\Pr[\text{red}] \leq \frac{2}{q} \text{ for } q \geq 2\Delta \dots$$



Local q -coloring Sampler

Colorings

$$\text{marginal } \Pr[\text{red}] = \begin{cases} 0 & \text{red} \in S_t \\ \frac{1}{|[q] \setminus S_t|} & \text{red} \notin S_t \end{cases}$$

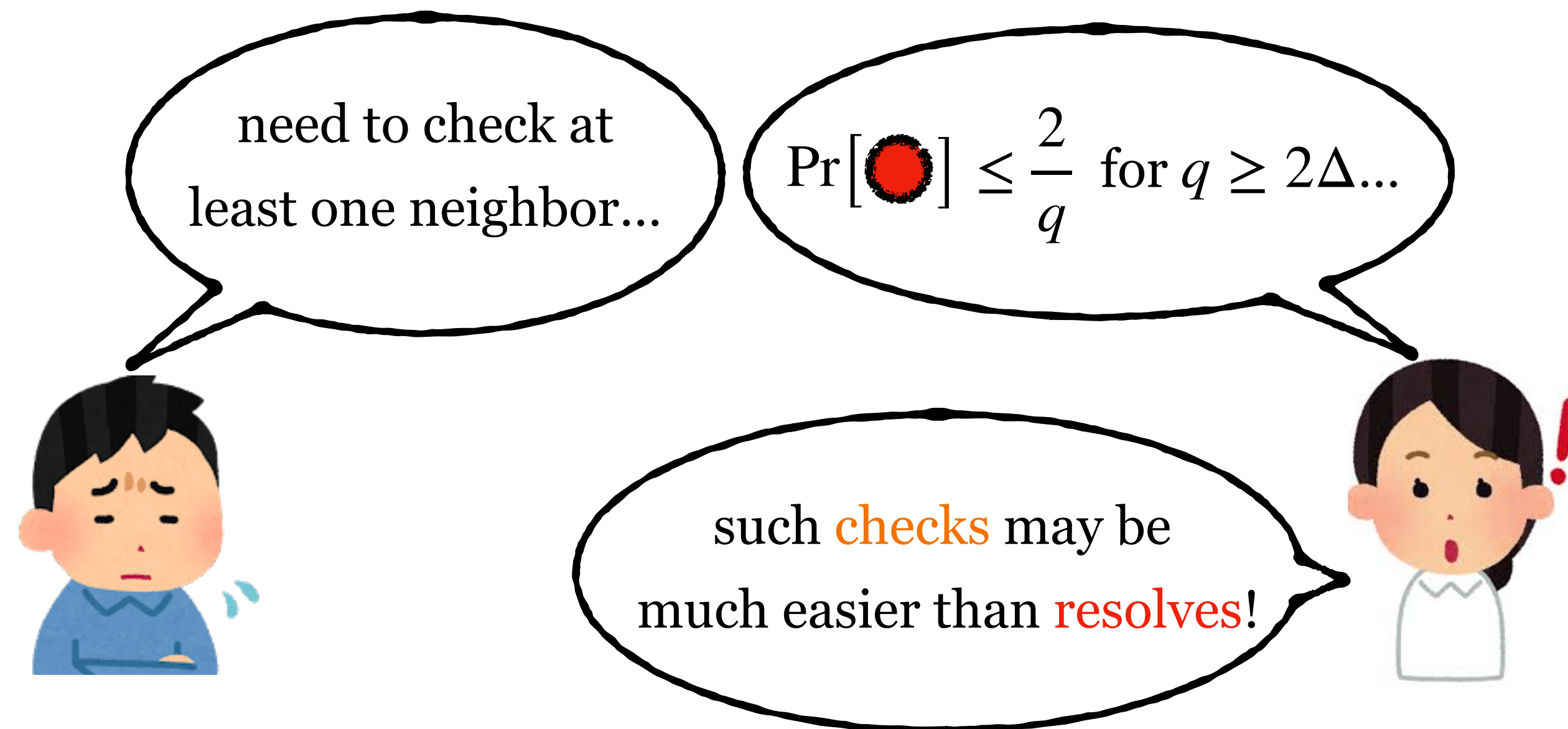
S_t = set of neighboring colors

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color  from $[q]$;

accept  if  $\notin S_t$;



Local q -coloring Sampler

Colorings

$$\text{marginal } \Pr[\bullet] = \begin{cases} 0 & \bullet \in S_t \\ \frac{1}{|[q] \setminus S_t|} & \bullet \notin S_t \end{cases}$$

S_t = set of neighboring colors

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color $\bullet$ from $[q]$;

accept $\bullet$ if $\bullet \notin S_t$;

Check if an update is $\bullet$

w.p. $1 - \frac{2}{q}$, directly certify that update is not $\bullet$;

otherwise:

check if $\bullet \in S_t$:

yes: certify that update is not $\bullet$;

no: certify that update: is $\bullet$ w.p. $\frac{q/2}{|[q] \setminus S_t|}$
is not $\bullet$ w.p. $\frac{|[q] \setminus S_t| - q/2}{|[q] \setminus S_t|}$

Local q -coloring Sampler

Colorings

$$\text{marginal } \Pr[\bullet] = \begin{cases} 0 & \bullet \in S_t \\ \frac{1}{|[q] \setminus S_t|} & \bullet \notin S_t \end{cases}$$

S_t = set of neighboring colors

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color $\bullet$ from $[q]$;

accept $\bullet$ if $\bullet \notin S_t$;

Check if an update is $\bullet$

w.p. $1 - \frac{2}{q}$, directly **certify** that update is not $\bullet$;

otherwise:

check if $\bullet \in S_t$:

yes: **certify** that update is not $\bullet$;

no: **certify** that update: is $\bullet$ w.p. $\frac{q/2}{|[q] \setminus S_t|}$
is not $\bullet$ w.p. $\frac{|[q] \setminus S_t| - q/2}{|[q] \setminus S_t|}$

How to **certify**?



Local q -coloring Sampler

Colorings

$$\text{marginal } \Pr[\bullet] = \begin{cases} 0 & \bullet \in S_t \\ \frac{1}{|[q] \setminus S_t|} & \bullet \notin S_t \end{cases}$$

S_t = set of neighboring colors

Check if an update is $\bullet$

w.p. $1 - \frac{2}{q}$, directly **certify** that update is not $\bullet$;
otherwise:

check if $\bullet \in S_t$:

yes: **certify** that update is not $\bullet$;

no: **certify** that update: is $\bullet$ w.p. $\frac{q/2}{|[q] \setminus S_t|}$
is not $\bullet$ w.p. $\frac{|[q] \setminus S_t| - q/2}{|[q] \setminus S_t|}$

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color $\bullet$ from $[q]$;

accept $\bullet$ if $\bullet \notin S_t$;

How to **certify**?

How to sample w.p.
 $\frac{q/2}{|[q] \setminus S_t|}$?



Local q -coloring Sampler

Resolve an update

for $i = 1, 2, \dots$:
propose a **uniform** color  from L_t ;
accept  if  $\notin S_t$;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly **certify** that update is not ;
otherwise:

check if  $\in S_t$:

yes: **certify** that update is not ;

no: **certify** that update: is  w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$
is not  w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

t	...	-8	-7	-6	-5	-4	-3	-2	-1	0
L_t	...	      	      	      	      	      	      	      	      	      

Local q -coloring Sampler

Resolve an update

for $i = 1, 2, \dots$:
propose a **uniform** color  from L_t ;
accept  if  $\notin S_t$;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly **certify** that update is not ;
otherwise:

check if  $\in S_t$:

yes: **certify** that update is not ;

no: **certify** that update: is  w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$
is not  w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

t	...	-8	-7	-6	-5	-4	-3	-2	-1	0
L_t	...	      	      	      	      	      	      	      	      	      

certify update is : $L_t \leftarrow \{\text{img alt="red circle" data-bbox="853 579 882 619"}\}$

certify update is not : $L_t \leftarrow L_t \setminus \{\text{img alt="red circle" data-bbox="902 634 931 674"}\}$

***will ensure that $|L_t| \geq 2\Delta$ for $|L_t| \neq 1$**

Local q -coloring Sampler

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color  from L_t ;

accept  if  $\notin S_t$;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly **certify** that update is not ;
otherwise:

check if  $\in S_t$:

yes: **certify** that update is not ;

no: **certify** that update: is  w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$
is not  w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

t	...	-8	-7	-6	-5	-4	-3	-2	-1	0
L_t	...	      	    	      	   	      		      	      	

certify update is : $L_t \leftarrow \{\text{img alt="red circle" data-bbox="854 583 884 623} \}$

certify update is not : $L_t \leftarrow L_t \setminus \{\text{img alt="red circle" data-bbox="904 639 934 679} \}$

***will ensure that $|L_t| \geq 2\Delta$ for $|L_t| \neq 1$**

Local q -coloring Sampler

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color  from L_t ;

accept  if  $\notin S_t$;

feels strange my state depends on **future** proposals...

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly **certify** that update is not ;
otherwise:

check if  $\in S_t$:

yes: **certify** that update is not ;

no: **certify** that update: is  w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$
is not  w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

t	...	-8	-7	-6	-5	-4	-3	-2	-1	0
L_t	...	      	    	      	   	      		      	      	

certify update is : $L_t \leftarrow \{\text{img alt="red circle" data-bbox="855 585 885 625"}\}$

certify update is not : $L_t \leftarrow L_t \setminus \{\text{img alt="red circle" data-bbox="905 640 935 680"}\}$

***will ensure that $|L_t| \geq 2\Delta$ for $|L_t| \neq 1$**

Local q -coloring Sampler

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color  from L_t ;

accept  if  $\notin S_t$;

feels strange my state depends on **future** proposals...

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly **certify** that update is not ;
otherwise:

check if  $\in S_t$:

yes: **certify** that update is not ;

no: **certify** that update: is  w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$
is not  w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

t	...	-8	-7	-6	-5	-4	-3	-2	-1	0
L_t	...	      	   	      	   	      		      	      	

certify update is : $L_t \leftarrow \{\text{img alt="red circle" data-bbox="858 582 882 616"}\}$

certify update is not : $L_t \leftarrow L_t \setminus \{\text{img alt="red circle" data-bbox="902 638 926 672"}\}$

***will ensure that $|L_t| \geq 2\Delta$ for $|L_t| \neq 1$**

non-Markovian grand coupling!

Local q -coloring Sampler

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color  from L_t ;

accept  if  $\notin S_t$;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly **certify** that update is not ;
otherwise:

check if  $\in S_t$:

yes: **certify** that update is not ;

no: **certify** that update: is  w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$
is not  w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

Local q -coloring Sampler

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color  from L_t ;

accept  if  $\notin S_t$;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly **certify** that update is not ;
otherwise:

check if  $\in S_t$:

yes: **certify** that update is not ;

no: **certify** that update: is  w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$
is not  w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

Explicitly finding S_t would be costly...



Local q -coloring Sampler

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color  from L_t ;

accept  if  $\notin S_t$;

an $\frac{|L_t \setminus S_t|}{|L_t|}$ -probability coin

propose a **uniform** color  from L_t ;

return  if  $\notin S_t$ and return  otherwise;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly **certify** that update is not ;
otherwise:

check if  $\in S_t$:

yes: **certify** that update is not ;

no: **certify** that update: is  w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$
is not  w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

Explicitly finding S_t would be costly...



Local q -coloring Sampler

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color  from L_t ;

accept  if  $\notin S_t$;

an $\frac{|L_t \setminus S_t|}{|L_t|}$ -probability coin

propose a **uniform** color  from L_t ;

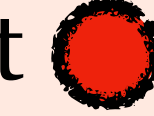
return  if  $\notin S_t$ and return  otherwise;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly **certify** that update is not ;
otherwise:

check if  $\in S_t$:

yes: **certify** that update is not ;

no: **certify** that update: is  w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$
is not  w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$



Explicitly finding S_t would be costly...

Local q -coloring Sampler

Resolve an update

for $i = 1, 2, \dots$:

propose a **uniform** color  from L_t ;

accept  if  $\notin S_t$;

an $\frac{|L_t \setminus S_t|}{|L_t|}$ -probability coin

propose a **uniform** color  from L_t ;

return  if  $\notin S_t$ and return  otherwise;

Check if an update is

w.p. $1 - \frac{2}{|L_t|}$, directly **certify** that update is not ;
otherwise:

check if  $\in S_t$:

yes: **certify** that update is not ;

no: **certify** that update: is  w.p. $\frac{|L_t|/2}{|L_t \setminus S_t|}$
is not  w.p. $\frac{|L_t \setminus S_t| - |L_t|/2}{|L_t \setminus S_t|}$

Efficiency Analysis

We carefully design a **potential function** Φ that

- $\Phi = \Phi(\Pi) \geq 0$ depends on the state of the algorithm;
- $\Phi = 0$ upon termination;
- Φ evolves as a **supermartingale** with bounded differences.



Summary

We propose

- **the first** local sampler for **near-critical** Ising model;
- **the first** local sampler for q -coloring (with $q = O(\Delta)$);

Our local samplers are **perfect**, runs in **expected linear time**, and is based on an extension of the **Coupling Towards The Past (CTTP)** framework.

Open problems

- Can local samplers achieve the exact **critical** thresholds/lower bounds ?
- Can local samplers achieve the thresholds for **perfect sampling**? (i.e., bounds achievable by Coupling From The Past method for q -colorings)
- More **applications** of local samplers?

Summary

We propose

- **the first** local sampler for **near-critical** Ising model;
- **the first** local sampler for q -coloring (with $q = O(\Delta)$);

Our local samplers are **perfect**, runs in **expected linear time**, and is based on an extension of the **Coupling Towards The Past (CTTP)** framework.

Thanks! Any questions?

Open problems

- Can local samplers achieve the exact **critical** thresholds/lower bounds ?
- Can local samplers achieve the thresholds for **perfect sampling**? (i.e., bounds achievable by Coupling From The Past method for q -colorings)
- More **applications** of local samplers?